# A SHORT TABLE OF INTEGRALS

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# A SHORT TABLE OF INTEGRALS

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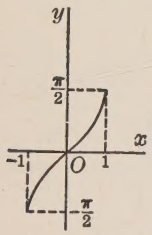
# TABLE OF INTEGRALS.

## PRINCIPAL VALUES.

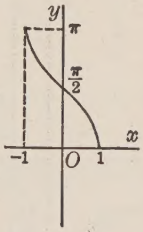
In the following tables the inverse trigonometric functions are to be understood as restricted to their *principal values*. These are indicated by the accompanying figures.

$y = \sin^{-1}x.$




$$-\frac{\pi}{2} \leqq \sin^{-1}x \leqq \frac{\pi}{2}.$$

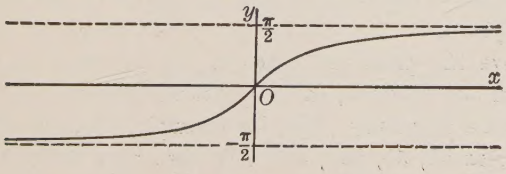
$y = \cos^{-1}x.$




$$0 \leqq \cos^{-1}x \leqq \pi.$$

The curves representing the functions $\tan^{-1}x$ and $\operatorname{ctn}^{-1}x$ extend indefinitely in both directions.

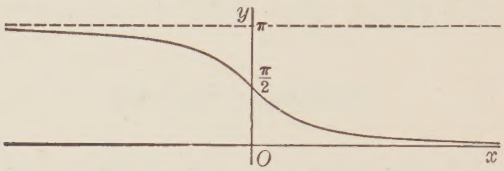
$y = \tan^{-1}x.$




$$-\frac{\pi}{2} < \tan^{-1}x < \frac{\pi}{2}.$$

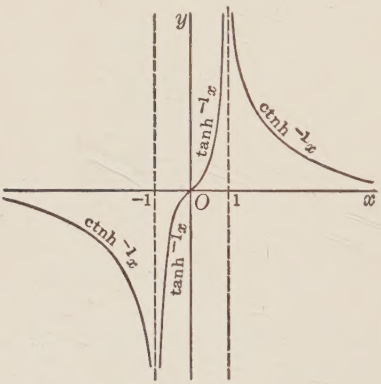
The principal value of $\operatorname{ctn}^{-1}x$ is connected with the principal value of $\tan^{-1}x$ by the relation $\tan^{-1}x + \operatorname{ctn}^{-1}x = \frac{1}{2}\pi$.

$y = \operatorname{ctn}^{-1}x.$

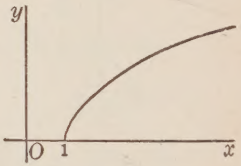



$0 < \operatorname{ctn}^{-1}x < \pi.$

The tables are adapted to the use of the hyperbolic functions, and graphs of three of them follow.




$y = \cosh^{-1}x.$




In certain trigonometric formulas, notably those in which the integration has been effected by means of the substitution $z = \tan\frac{1}{2}x$, there is a hidden use of the principal value, over and above the principal value of the function occurring explicitly in the formula, and so restrictions on the independent variable are necessary. See, for example, Formula 300.

Formulas 49, 50, 298, and 300 have been recast to the end that they be correct for all values of $a$, $b$ for which they have a meaning, that they cover all cases, and that they be better

adapted to computation. Only one formula, 316, has been dropped, as being both incomplete and unnecessary; and the numbering of the formulas has been retained except in the case of Formulas 314–316.

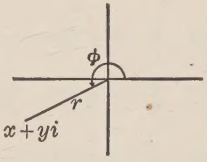
The formula

$$\log(x+yi) = \tfrac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x}$$

is treacherous, since the values of the multiple-valued function on the left cannot be expressed in terms of the principal value of $\tan^{-1}y/x$, $\pm k\pi$. Sometimes an even multiple of $\pi$ must be added, and sometimes an odd multiple. The formula which is correct in all cases is the following:

$$\log(x+yi) = \log r + \phi i,$$

$$x = r\cos\phi, \quad y = r\sin\phi, \quad r = \sqrt{x^2+y^2}.$$




The tables of tabulated functions remain as in the earlier edition, except that the pages of hyperbolic functions have been revised and a table of square roots has been added.

## I. FUNDAMENTAL FORMS.

1. $\int a\,dx = ax.$

2. $\int af(x)\,dx = a\int f(x)\,dx.$

3. $\int \frac{dx}{x} = \log x.$ $[\log x = \log(-x) + (2k+1)\pi i.]$

4. $\int x^m dx = \frac{x^{m+1}}{m+1}$, when $m$ is different from $-1$.

5. $\int e^x dx = e^x.$

6. $\int a^x \log a\,dx = a^x.$

7. $\int \frac{dx}{1+x^2} = \tan^{-1}x$, or $-\operatorname{ctn}^{-1}x$.

8. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x$, or $-\cos^{-1}x$.

9. $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}x$, or $-\operatorname{csc}^{-1}x$.

10. $\int \frac{dx}{\sqrt{2x-x^2}} = \operatorname{versin}^{-1}x$, or $-\operatorname{coversin}^{-1}x$.

11. $\int \cos x\,dx = \sin x$, or $-\operatorname{coversin} x$.

12. $\int \sin x\,dx = -\cos x$, or $\operatorname{versin} x$.

13. $\int \operatorname{ctn} x\,dx = \log \sin x$.

14. $\int \tan x\,dx = -\log \cos x$.

15. $\int \tan x \sec x\,dx = \sec x$.

16. $\int \sec^2 x\,dx = \tan x$.

17. $\int \operatorname{csc}^2 x\,dx = -\operatorname{ctn} x$.

In the following formulas, $u$, $v$, $w$, and $y$ represent any functions of $x$:

18. $\int (u+v+w+\text{etc.})\,dx = \int u\,dx + \int v\,dx + \int w\,dx + \text{etc.}$

19*a*. $\int u\,dv = uv - \int v\,du$.

19*b*. $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{au}{dx}\,dx$.

20. $\int f(y)\,dx = \int \frac{f(y)\,dy}{\frac{dy}{dx}}$.

## II. RATIONAL ALGEBRAIC FUNCTIONS.

### A. — Expressions Involving $(a + bx)$.

The substitution of $y$ or $z$ for $x$, where $y \equiv a + bx$, $z \equiv (a + bx)/x$, gives

**21.** $\int (a + bx)^m dx = \frac{1}{b} \int y^m dy.$

**22.** $\int x (a + bx)^m dx = \frac{1}{b^2} \int y^m (y - a) dy.$

**23.** $\int x^n (a + bx)^m dx = \frac{1}{b^{n+1}} \int y^m (y - a)^n dy.$

**24.** $\int \frac{x^n dx}{(a + bx)^m} = \frac{1}{b^{n+1}} \int \frac{(y - a)^n dy}{y^m}.$

**25.** $\int \frac{dx}{x^n (a + bx)^m} = -\frac{1}{a^{m+n-1}} \int \frac{(z - b)^{m+n-2} dz}{z^m}.$

Whence

**26.** $\int \frac{dx}{a + bx} = \frac{1}{b} \log (a + bx).$

**27.** $\int \frac{dx}{(a + bx)^2} = -\frac{1}{b (a + bx)}.$

**28.** $\int \frac{dx}{(a + bx)^3} = -\frac{1}{2 b (a + bx)^2}.$

**29.** $\int \frac{x dx}{a + bx} = \frac{1}{b^2} [a + bx - a \log (a + bx)].$

**30.** $\int \frac{x dx}{(a + bx)^2} = \frac{1}{b^2} \left[ \log (a + bx) + \frac{a}{a + bx} \right].$

31. $$\int \frac{x\,dx}{(a+bx)^3} = \frac{1}{b^2}\left[-\frac{1}{a+bx} + \frac{a}{2(a+bx)^2}\right].$$

32. $$\int \frac{x^2\,dx}{a+bx} = \frac{1}{b^3}\left[\tfrac{1}{2}(a+bx)^2 - 2a(a+bx) + a^2\log(a+bx)\right].$$

33. $$\int \frac{x^2\,dx}{(a+bx)^2} = \frac{1}{b^3}\left[a+bx-2a\log(a+bx)-\frac{a^2}{a+bx}\right].$$

34. $$\int \frac{dx}{x(a+bx)} = -\frac{1}{a}\log\frac{a+bx}{x}.$$ *

35. $$\int \frac{dx}{x(a+bx)^2} = \frac{1}{a(a+bx)} - \frac{1}{a^2}\log\frac{a+bx}{x}.$$

36. $$\int \frac{(a+bx)\,dx}{a'+b'x} = \frac{bx}{b'} + \frac{ab'-a'b}{b'^2}\log(a'+b'x).$$

37. $$\int (a+bx)^n(a'+b'x)^m\,dx = \frac{1}{(m+n+1)\,b}\Big((a+bx)^{n+1}(a'+b'x)^m - m(ab'-a'b)\int (a+bx)^n(a'+b'x)^{m-1}\,dx\Big).$$

38. $$\int \frac{(a+bx)^n\,dx}{(a'+b'x)^m} = -\frac{1}{(m-1)(ab'-a'b)}\left(\frac{(a+bx)^{n+1}}{(a'+b'x)^{m-1}} + (m-n-2)\,b\int\frac{(a+bx)^n\,dx}{(a'+b'x)^{m-1}}\right)$$

$$= -\frac{1}{(m-n-1)\,b'}\left(\frac{(a+bx)^n}{(a'+b'x)^{m-1}} + n(ab'-a'b)\int\frac{(a+bx)^{n-1}\,dx}{(a'+b'x)^m}\right)$$

$$= -\frac{1}{(m-1)\,b'}\left(\frac{(a+bx)^n}{(a'+b'x)^{m-1}} - nb\int\frac{(a+bx)^{n-1}\,dx}{(a'+b'x)^{m-1}}\right).$$

* $\int \frac{dx}{x^2(a+bx)} = -\frac{1}{ax} + \frac{b}{a^2}\log\frac{a+bx}{x}.$

39. $$\int \frac{dx}{(a+bx)(a'+b'x)} = \frac{1}{ab'-a'b} \cdot \log \frac{a'+b'x}{a+bx}.$$

40. $$\int \frac{dx}{(a+bx)^n (a'+b'x)^m}$$
$$= \frac{1}{(m-1)(ab'-a'b)} \left( \frac{-1}{(a+bx)^{n-1}(a'+b'x)^{m-1}} \right.$$
$$\left. -(m+n-2)\, b \int \frac{dx}{(a+bx)^n (a'+b'x)^{m-1}} \right).$$

41. $$\int \frac{x\,dx}{(a+bx)(a'+b'x)}$$
$$= \frac{1}{ab'-a'b} \left( \frac{a}{b} \log (a+bx) - \frac{a'}{b'} \log (a'+b'x) \right).$$

42. $$\int \frac{dx}{(a+bx)^2 (a'+b'x)}$$
$$= \frac{1}{ab'-a'b} \left( \frac{1}{a+bx} + \frac{b'}{ab'-a'b} \log \frac{a'+b'x}{a+bx} \right).$$

43. $$\int \frac{x\,dx}{(a+bx)^2 (a'+b'x)}$$
$$= \frac{-a}{b(ab'-a'b)(a+bx)} - \frac{a'}{(ab'-a'b)^2} \log \frac{a'+b'x}{a+bx}.$$

44. $$\int \frac{x^2\,dx}{(a+bx)^2 (a'+b'x)} = \frac{a^2}{b^2 (ab'-a'b)(a+bx)}$$
$$+ \frac{1}{(ab'-a'b)^2} \left[ \frac{a'^2}{b'} \log (a'+b'x) + \frac{a(ab'-2a'b)}{b^2} \log (a+bx) \right].$$

45. $$\int (a+bx)^{\frac{1}{n}} dx = \frac{n}{(n+1)b} (a+bx)^{\frac{n+1}{n}}.$$

46. $$\int \frac{dx}{(a+bx)^{\frac{1}{n}}} = \frac{n}{(n-1)b} (a+bx)^{\frac{n-1}{n}}.$$

## B. — Expressions Involving $(a + bx^n)$.

**47.** $\int \frac{dx}{c^2 + x^2} = \frac{1}{c} \tan^{-1} \frac{x}{c} = \frac{1}{c} \sin^{-1} \frac{x}{\sqrt{x^2 + c^2}}.$

**48.** $\int \frac{dx}{c^2 - x^2} = \frac{1}{2c} \log \frac{c + x}{c - x} = \frac{1}{c} \tanh^{-1} \frac{x}{c}$, or $\frac{1}{c} \operatorname{ctnh}^{-1} \frac{x}{c}.$

**49.** $\int \frac{dx}{a + bx^2} = \frac{1}{\sqrt{ab}} \tan^{-1} \frac{x\sqrt{ab}}{a}.$

**50.** $\int \frac{dx}{a + bx^2} = \frac{1}{2\sqrt{-ab}} \log \frac{a + x\sqrt{-ab}}{a - x\sqrt{-ab}},$

or $\frac{1}{\sqrt{-ab}} \tanh^{-1} \frac{x\sqrt{-ab}}{a}$, or $\frac{1}{\sqrt{-ab}} \operatorname{ctnh}^{-1} \frac{x\sqrt{-ab}}{a}.$

**51.** $\int \frac{dx}{(a + bx^2)^2} = \frac{x}{2a(a + bx^2)} + \frac{1}{2a} \int \frac{dx}{a + bx^2}.$

**52.** $\int \frac{dx}{(a + bx^2)^{m+1}} = \frac{1}{2ma} \cdot \frac{x}{(a + bx^2)^m} + \frac{2m - 1}{2ma} \int \frac{dx}{(a + bx^2)^m}.$

**53.** $\int \frac{x\,dx}{a + bx^2} = \frac{1}{2b} \log \left( x^2 + \frac{a}{b} \right).$

**54.** $\int \frac{x\,dx}{(a + bx^2)^{m+1}} = \frac{1}{2} \int \frac{dz}{(a + bz)^{m+1}}$, where $z = x^2.$

**55.** $\int \frac{dx}{x(a + bx^2)} = \frac{1}{2a} \log \frac{x^2}{a + bx^2}.$

**56.** $\int \frac{x^2\,dx}{a + bx^2} = \frac{x}{b} - \frac{a}{b} \int \frac{dx}{a + bx^2}.$

**57.** $\int \frac{dx}{x^2(a + bx^2)} = -\frac{1}{ax} - \frac{b}{a} \int \frac{dx}{a + bx^2}.$

**58.** $\int \frac{x^2\,dx}{(a + bx^2)^{m+1}} = \frac{-x}{2mb(a + bx^2)^m} + \frac{1}{2mb} \int \frac{dx}{(a + bx^2)^m}.$

**59.** $\int \frac{dx}{x^2(a + bx^2)^{m+1}} = \frac{1}{a} \int \frac{dx}{x^2(a + bx^2)^m} - \frac{b}{a} \int \frac{dx}{(a + bx^2)^{m+1}}.$

**60.** $\int \frac{dx}{a+bx^3} = \frac{k}{3a}\left[\frac{1}{2}\log\left(\frac{(k+x)^2}{k^2-kx+x^2}\right)+\sqrt{3}\tan^{-1}\frac{2x-k}{k\sqrt{3}}\right]$, where $bk^3 = a$.

**61.** $\int \frac{x\,dx}{a+bx^3} = \frac{1}{3bk}\left[\frac{1}{2}\log\left(\frac{k^2-kx+x^2}{(k+x)^2}\right)+\sqrt{3}\tan^{-1}\frac{2x-k}{k\sqrt{3}}\right]$, where $bk^3 = a$.

**62.** $\int \frac{dx}{x(a+bx^n)} = \frac{1}{an}\log\frac{x^n}{a+bx^n}$.

**63.** $\int \frac{dx}{(a+bx^n)^{m+1}} = \frac{1}{a}\int \frac{dx}{(a+bx^n)^m} - \frac{b}{a}\int \frac{x^n\,dx}{(a+bx^n)^{m+1}}$.

**64.** $\int \frac{x^m\,dx}{(a+bx^n)^{p+1}} = \frac{1}{b}\int \frac{x^{m-n}\,dx}{(a+bx^n)^p} - \frac{a}{b}\int \frac{x^{m-n}\,dx}{(a+bx^n)^{p+1}}$.

**65.** $\int \frac{dx}{x^m(a+bx^n)^{p+1}} = \frac{1}{a}\int \frac{dx}{x^m(a+bx^n)^p} - \frac{b}{a}\int \frac{dx}{x^{m-n}(a+bx^n)^{p+1}}$.

**66.** $\int x^{m-1}(a+bx^n)^p\,dx =$

$$\begin{cases} \frac{1}{b(m+np)}\left[x^{m-n}(a+bx^n)^{p+1} - (m-n)a\int x^{m-n-1}(a+bx^n)^p\,dx\right]. \\ \frac{1}{m+np}\left[x^m(a+bx^n)^p + npa\int x^{m-1}(a+bx^n)^{p-1}\,dx\right]. \\ \frac{1}{ma}\left[x^m(a+bx^n)^{p+1} - (m+np+n)b\int x^{m+n-1}(a+bx^n)^p\,dx\right]. \\ \frac{1}{an(p+1)}\left[-x^m(a+bx^n)^{p+1} + (m+np+n)\int x^{m-1}(a+bx^n)^{p+1}\,dx\right]. \end{cases}$$

## C. — Expressions Involving $(a + bx + cx^2)$.

Let $X = a + bx + cx^2$ and $q = 4ac - b^2$, then

67. $$\int \frac{dx}{X} = \frac{2}{\sqrt{q}} \tan^{-1} \frac{2cx + b}{\sqrt{q}}.$$

68. $$\int \frac{dx}{X} = \frac{1}{\sqrt{-q}} \log \frac{2cx + b - \sqrt{-q}}{2cx + b + \sqrt{-q}},$$

or $$\frac{-2}{\sqrt{-q}} \tanh^{-1} \frac{2cx + b}{\sqrt{-q}}, \text{ or } \frac{-2}{\sqrt{-q}} \operatorname{ctnh}^{-1} \frac{2cx + b}{\sqrt{-q}}$$

69. $$\int \frac{dx}{X^2} = \frac{2cx + b}{qX} + \frac{2c}{q} \int \frac{dx}{X}.$$

70. $$\int \frac{dx}{X^3} = \frac{2cx + b}{q} \left( \frac{1}{2X^2} + \frac{3c}{qX} \right) + \frac{6c^2}{q^2} \int \frac{dx}{X}.$$

71. $$\int \frac{dx}{X^{n+1}} = \frac{2cx + b}{nqX^n} + \frac{2(2n-1)c}{qn} \int \frac{dx}{X^n}.$$

72. $$\int \frac{x\,dx}{X} = \frac{1}{2c} \log X - \frac{b}{2c} \int \frac{dx}{X}.$$

73. $$\int \frac{x\,dx}{X^2} = -\frac{bx + 2a}{qX} - \frac{b}{q} \int \frac{dx}{X}.$$

74. $$\int \frac{x\,dx}{X^{n+1}} = -\frac{2a + bx}{nqX^n} - \frac{b(2n-1)}{nq} \int \frac{dx}{X^n}.$$

75. $$\int \frac{x^2}{X} dx = \frac{x}{c} - \frac{b}{2c^2} \log X + \frac{b^2 - 2ac}{2c^2} \int \frac{dx}{X}.$$

76. $$\int \frac{x^2}{X^2} dx = \frac{(b^2 - 2ac)x + ab}{cqX} + \frac{2a}{q} \int \frac{dx}{X}.$$

77. $$\int \frac{x^m dx}{X^{n+1}} = -\frac{x^{m-1}}{(2n - m + 1)cX^n} - \frac{n - m + 1}{2n - m + 1} \cdot \frac{b}{c} \int \frac{x^{m-1} dx}{X^{n+1}} + \frac{m - 1}{2n - m + 1} \cdot \frac{a}{c} \int \frac{x^{m-2} dx}{X^{n+1}}.$$

78. $$\int \frac{dx}{xX} = \frac{1}{2a} \log \frac{x^2}{X} - \frac{b}{2a} \int \frac{dx}{X}.$$

79. $$\int \frac{dx}{x^2 X} = \frac{b}{2a^2} \log \frac{X}{x^2} - \frac{1}{ax} + \left( \frac{b^2}{2a^2} - \frac{c}{a} \right) \int \frac{dx}{X}.$$

80. $$\int \frac{dx}{x^m X^{n+1}} = -\frac{1}{(m-1)ax^{m-1}X^n} - \frac{n+m-1}{m-1} \cdot \frac{b}{a} \int \frac{dx}{x^{m-1}X^{n+1}} - \frac{2n+m-1}{m-1} \cdot \frac{c}{a} \int \frac{dx}{x^{m-2}X^{n+1}}.$$

81. $$\int X^n dx = \frac{1}{2(2n+1)c} \left( (b+2cx) X^n + nq \int X^{n-1} dx \right).$$

82. $$\int \frac{dx}{x X^n} = \frac{1}{2a(n-1)X^{n-1}} - \frac{b}{2a} \int \frac{dx}{X^n} + \frac{1}{a} \int \frac{dx}{x X^{n-1}}.$$

83. $$\int \frac{dx}{(a'+b'x)X} = \frac{1}{2(ab'^2 - a'bb' + a'^2c)} \left( b' (\log (a'+b'x)^2 - \log X) + (2a'c - bb') \int \frac{dx}{X} \right).$$

84. $$\int (a'+b'x) X^n dx = \frac{b' X^{n+1}}{2(n+1)c} + \frac{2a'c - bb'}{2c} \int X^n dx.$$

85. $$\int \frac{(a'+b'x)dx}{X^n} = -\frac{b'}{2(n-1)cX^{n-1}} + \frac{2a'c - bb'}{2c} \int \frac{dx}{X^n}.$$

86. $$\int (a'+b'x)^m X^n dx = \frac{1}{(m+2n+1)c} \left( b'(a'+b'x)^{m-1} X^{n+1} + (m+n)(2a'c - bb') \int (a'+b'x)^{m-1} X^n dx - (m-1)(ab'^2 - a'bb' + ca'^2) \int (a'+b'x)^{m-2} X^n dx \right).$$

87. $$\int \frac{(a'+b'x)^m\,dx}{X^n} = \frac{1}{q\,(n-1)}\left(\frac{(b+2\,cx)(a'+b'x)^m}{X^{n-1}}\right.$$

$$-2\,(m-2\,n+3)\,c\int\frac{(a'+b'x)^m\,dx}{X^{n-1}}$$

$$\left.+\,m\,(2\,a'c-bb')\int\frac{(a'+b'x)^{m-1}\,dx}{X^{n-1}}\right)$$

$$=\frac{1}{(m-2\,n+1)\,c}\left(\frac{b'\,(a'+b'x)^{m-1}}{X^{n-1}}\right.$$

$$+\,(m-n)\,(2\,a'c-bb')\int\frac{(a'+b'x)^{m-1}\,dx}{X^n}$$

$$\left.-\,(m-1)\,(ab'^2-a'bb'+ca'^2)\int\frac{(a'+b'x)^{m-2}\,dx}{X^n}\right).$$

88. $$\int\frac{X^n\,dx}{(a'+b'x)^m}$$

$$=\frac{1}{b'^2\,(m-1)}\left(\frac{-\,b'X^n}{(a'+b'x)^{m-1}}\right.$$

$$+\,n\,(bb'-2\,a'c)\int\frac{X^{n-1}dx}{(a'+b'x)^{m-1}}$$

$$\left.+\,2\,nc\int\frac{X^{n-1}\,dx}{(a'+b'x)^{m-2}}\right)$$

$$=-\frac{1}{(m-2\,n-1)\,b'^2}\left(\frac{+\,b'X^n}{(a'+b'x)^{m-1}}\right.$$

$$+\,2\,n\,(ab'^2-a'bb'+ca'^2)\int\frac{X^{n-1}\,dx}{(a'+b'x)^m}$$

$$\left.+\,n\,(bb'-2\,a'c)\int\frac{X^{n-1}\,dx}{(a'+b'x)^{m-1}}\right).$$

89. $$\int \frac{dx}{(a'+b'x)^m X^n}$$

$$= -\frac{1}{(m-1)(ab'^2 - a'bb' + ca'^2)}\left(\frac{b'}{(a'+b'x)^{m-1}X^{n-1}}\right.$$
$$+ (m+n-2)(bb'-2ca')\int \frac{dx}{(a'+b'x)^{m-1}X^n}$$
$$\left. + (m+2n-3)c\int \frac{dx}{(a'+b'x)^{m-2}X^n}\right)$$

$$= \frac{1}{2(ab'^2 - a'bb' + ca'^2)}\left(\frac{b'}{(n-1)(a'+b'x)^{m-1}X^{n-1}}\right.$$
$$+ (2a'c - bb')\int \frac{dx}{(a'+b'x)^{m-1}X^n}$$
$$\left. + \frac{(m+2n-3)b'^2}{n-1}\int \frac{dx}{(a'+b'x)^m X^{n-1}}\right).$$

If $ab'^2 - a'bb' + ca'^2 = 0$,

$$\int \frac{dx}{(a'+b'x)^m X^n}$$
$$= \frac{-1}{(m+n-1)(bb'-2a'c)}\left(\frac{b'}{(a'+b'x)^m X^{n-1}}\right.$$
$$\left. + (m+2n-2)c\int \frac{dx}{(a'+b'x)^{m-1}X^n}\right).$$

### D. — Rational Fractions.

Every proper fraction can be represented by the general form:

$$\frac{f(x)}{F(x)} = \frac{g_1 x^{n-1} + g_2 x^{n-2} + g_3 x^{n-3} + \cdots + g_n}{x^n + k_1 x^{n-1} + k_2 x^{n-2} + \cdots + k_n}.$$

If $a$, $b$, $c$, etc., are the roots of the equation $F(x) = 0$, so that

$$F(x) = (x-a)^p (x-b)^q (x-c)^r \cdots,$$

then

$$\frac{f(x)}{F(x)} = \frac{A_1}{(x-a)^p} + \frac{A_2}{(x-a)^{p-1}} + \frac{A_3}{(x-a)^{p-2}} + \cdots + \frac{A_p}{x-a}$$
$$+ \frac{B_1}{(x-b)^q} + \frac{B_2}{(x-b)^{q-1}} + \frac{B_3}{(x-b)^{q-2}} + \cdots + \frac{B^q}{x-b}$$
$$+ \frac{C_1}{(x-c)^r} + \frac{C_2}{(x-c)^{r-1}} + \frac{C_3}{(x-c)^{r-2}} + \cdots + \frac{C_r}{x-c}$$
$$+ \quad \cdots \qquad \cdots \qquad \cdots \qquad \cdots \qquad \cdots,$$

where the numerators of the separate fractions may be determined by the equations

$$A_m = \frac{\phi_1{}^{[m-1]}(a)}{(m-1)!}, \quad B_m = \frac{\phi_2{}^{[m-1]}(b)}{(m-1)!} \quad \text{etc., etc.}$$

$$\phi_1(x) = \frac{f(x)\,(x-a)^p}{F(x)}, \quad \phi_2(x) = \frac{f(x)\,(x-b)^q}{F(x)}, \quad \text{etc., etc.}$$

If $a$, $b$, $c$, etc., are single roots, then $p = q = r = \cdots = 1$, and

$$\frac{f(x)}{F(x)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} \cdots$$

where $$A = \frac{f(a)}{F'(a)}, \quad B = \frac{f(b)}{F'(b)}, \quad \text{etc.}$$

The simpler fractions, into which the original fraction is thus divided, may be integrated by means of the formulas:

**90.** $$\int \frac{h\,dx}{(mx+n)^l} = \int \frac{h\,d(mx+n)}{m\,(mx+n)^l} = \frac{h}{m\,(1-l)\,(mx+n)^{l-1}},$$

and $$\int \frac{h\,dx}{mx+n} = \frac{h}{m} \log (mx+n).$$

If any of the roots of the equation $f(x) = 0$ are imaginary, the parts of the integral which arise from conjugate roots can be combined and the integral brought into a real form. The following formula, in which $i = \sqrt{-1}$, is often useful in combining logarithms of conjugate complex quantities:

$$\log(x \pm yi) = \tfrac{1}{2}\log(x^2 + y^2) \pm i \tan^{-1}\frac{y}{x}.$$

The identities given below are sometimes convenient:

$$\frac{1}{(a + bx^2)(a' + b'x^2)} \equiv \frac{1}{a'b - ab'} \cdot \left[\frac{b}{a + bx^2} - \frac{b'}{a' + b'x^2}\right],$$

$$\frac{m + nx}{(k + lx)(a + bx + cx^2)} \equiv \frac{1}{al^2 + ck^2 - bkl} \cdot \left[\frac{l(ml - nk)}{k + lx} + \frac{c(nk - ml)x + (aln + ckm - blm)}{a + bx + cx^2}\right],$$

$$\frac{l + mx^n}{(a + bx^n)(a' + b'x^n)} \equiv \frac{1}{a'b - ab'} \cdot \left[\frac{bl - am}{a + bx^n} + \frac{a'm - b'l}{a' + b'x^n}\right].$$

$$\frac{1}{(x + a)(x + b)(x + c)} = \frac{A}{x + a} + \frac{B}{x + b} + \frac{C}{x + c},$$

where

$$A = \frac{1}{(a - b)(a - c)}, \quad B = \frac{1}{(b - c)(b - a)}, \quad C = \frac{1}{(c - a)(c - b)}.$$

$$\frac{1}{(x + a)(x + b)(x + c)(x + g)} = \frac{A}{x + a} + \frac{B}{x + b} + \frac{C}{x + c} + \frac{G}{x + g},$$

where

$$A = \frac{1}{(b - a)(c - a)(g - a)}, \quad B = \frac{1}{(a - b)(c - b)(g - b)}, \text{ etc.}$$

## III. IRRATIONAL ALGEBRAIC FUNCTIONS.

### A. — Expressions Involving $\sqrt{a+bx}$.

The substitution of a new variable of integration, $y = \sqrt{a+bx}$, gives

**91.** $\int \sqrt{a+bx}\,dx = \frac{2}{3b}\sqrt{(a+bx)^3}.$

**92.** $\int x\sqrt{a+bx}\,dx = -\frac{2(2a-3bx)\sqrt{(a+bx)^3}}{15b^2}.$

**93.** $\int x^2\sqrt{a+bx}\,dx = \frac{2(8a^2-12abx+15b^2x^2)\sqrt{(a+bx)^3}}{105b^3}.$

**94.** $\int \frac{\sqrt{a+bx}}{x}dx = 2\sqrt{a+bx} + a\int \frac{dx}{x\sqrt{a+bx}}.$

**95.** $\int \frac{dx}{\sqrt{a+bx}} = \frac{2\sqrt{a+bx}}{b}.$

**96.** $\int \frac{x\,dx}{\sqrt{a+bx}} = -\frac{2(2a-bx)}{3b^2}\sqrt{a+bx}.$

**97.** $\int \frac{x^2dx}{\sqrt{a+bx}} = \frac{2(8a^2-4abx+3b^2x^2)}{15b^3}\sqrt{a+bx}.$

**98.** $\int \frac{dx}{x\sqrt{a+bx}} = \frac{1}{\sqrt{a}}\log\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}},$

or $\frac{-2}{\sqrt{a}}\tanh^{-1}\frac{\sqrt{a+bx}}{\sqrt{a}}$, or $\frac{-2}{\sqrt{a}}\operatorname{ctnh}^{-1}\frac{\sqrt{a+bx}}{\sqrt{a}}.$

**99.** $\int \frac{dx}{x\sqrt{a+bx}} = \frac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a+bx}{-a}}.$

100. $$\int \frac{dx}{x^2\sqrt{a+bx}} = -\frac{\sqrt{a+bx}}{ax} - \frac{b}{2a}\int \frac{dx}{x\sqrt{a+bx}}.$$

101. $$\int (a+bx)^{\pm\frac{n}{2}}\,dx = \frac{2}{b}\int y^{1\pm n}\,dy = \frac{2\,(a+bx)^{\frac{2\pm n}{2}}}{b\,(2\pm n)}.$$

102. $$\int x\,(a+bx)^{\pm\frac{n}{2}}\,dx = \frac{2}{b^2}\left[\frac{(a+bx)^{\frac{4\pm n}{2}}}{4\pm n} - \frac{a\,(a+bx)^{\frac{2\pm n}{2}}}{2\pm n}\right].$$

103. $$\int \frac{x^m\,dx}{\sqrt{a+bx}} = \frac{2\,x^m\sqrt{a+bx}}{(2\,m+1)\,b} - \frac{2\,ma}{(2\,m+1)\,b}\int \frac{x^{m-1}\,dx}{\sqrt{a+bx}}.$$

104. $$\int \frac{dx}{x^n\sqrt{a+bx}} = -\frac{\sqrt{a+bx}}{(n-1)ax^{n-1}} - \frac{(2n-3)b}{(2n-2)a}\int \frac{dx}{x^{n-1}\sqrt{a+bx}}$$

105. $$\int \frac{(a+bx)^{\frac{n}{2}}\,dx}{x} = b\int (a+bx)^{\frac{n-2}{2}}\,dx + a\int \frac{(a+bx)^{\frac{n-2}{2}}}{x}\,dx.$$

106. $$\int \frac{dx}{x\,(a+bx)^{\frac{m}{2}}} = \frac{1}{a}\int \frac{dx}{x\,(a+bx)^{\frac{m-2}{2}}} - \frac{b}{a}\int \frac{dx}{(a+bx)^{\frac{m}{2}}}.$$

107. $$\int f(x,\ \sqrt[n]{a+bx})\,dx = \frac{n}{b}\int f\left(\frac{z^n-a}{b},\ z\right) z^{n-1}\,dz,$$

where $z^n = a+bx$.

108. $$\int (a+bx)^{\frac{m}{n}}\,dx = \frac{n\,(a+bx)^{\frac{m+n}{n}}}{b\,(m+n)}.$$

109. $$\int f(x,\ (a+bx)^{\frac{m}{n}},\ (a+bx)^{\frac{p}{q}},\ \cdots)\,dx$$
$$= \frac{s}{b}\int f\left(\frac{y^s-a}{b},\ y^{\frac{ms}{n}},\ y^{\frac{ps}{q}},\ \cdots\right) y^{s-1}\,dy,$$

where $y^s = a+bx$, and $s$ is the least common multiple of $n$, $q$, etc.

B.—Expressions Involving Both $\sqrt{a+bx}$ and $\sqrt{a'+b'x}$.

Let $u = a + bx$, $v = a' + b'x$, and $k = ab' - a'b$, then

110. $$\int \sqrt{uv}\,dx = \frac{k + 2bv}{4bb'}\sqrt{uv} - \frac{k^2}{8bb'}\int \frac{dx}{\sqrt{uv}}.$$

111. $$\int \frac{\sqrt{v}\,dx}{\sqrt{u}} = \frac{1}{b}\sqrt{uv} - \frac{k}{2b}\int \frac{dx}{\sqrt{uv}}.$$

112. $$\int \frac{x\,dx}{\sqrt{uv}} = \frac{\sqrt{uv}}{bb'} - \frac{ab' + a'b}{2bb'}\int \frac{dx}{\sqrt{uv}}.$$

113. $$\int \frac{dx}{\sqrt{uv}} = \frac{2}{\sqrt{bb'}}\log\left(\sqrt{bb'u} + b\sqrt{v}\right)$$
$$= \frac{2}{\sqrt{-bb'}}\tan^{-1}\sqrt{\frac{-b'u}{bv}}, \text{ or } \frac{2}{\sqrt{bb'}}\tanh^{-1}\sqrt{\frac{b'u}{bv}}$$
$$= \frac{1}{\sqrt{-bb'}}\sin^{-1}\frac{2bb'x + a'b + ab'}{k}.$$

114. $$\int \frac{dx}{v\sqrt{u}} = \frac{1}{\sqrt{kb'}}\log\frac{b'\sqrt{u} - \sqrt{kb'}}{b'\sqrt{u} + \sqrt{kb'}} = \frac{2}{\sqrt{-kb'}}\tan^{-1}\frac{b'\sqrt{u}}{\sqrt{-kb'}}.$$

115. $$\int \frac{dx}{v\sqrt{uv}} = -\frac{2\sqrt{u}}{k\sqrt{v}}.$$

116. $$\int v^n\sqrt{u}\,dx = \frac{1}{(2m+3)b'}\left(2v^{n+1}\sqrt{u} + k\int \frac{v^n\,dx}{\sqrt{u}}\right).$$

117. $$\int \frac{\sqrt{u}\,dx}{v^m} = -\frac{1}{(2m-3)b'}\left(\frac{2\sqrt{u}}{v^{m-1}} + k\int \frac{dx}{v^m\sqrt{u}}\right)$$
$$= \frac{1}{(m-1)b'}\left(-\frac{\sqrt{u}}{v^{m-1}} + \tfrac{1}{2}b\int \frac{dx}{v^{m-1}\sqrt{u}}\right).$$

118. $$\int \frac{v^n\,dx}{\sqrt{u}} = \frac{2}{(2m+1)b}\left(v^n\sqrt{u} - nk\int \frac{v^{n-1}\,dx}{\sqrt{u}}\right).$$

**119.** $$\int \frac{dx}{v^m\sqrt{u}} = -\frac{1}{(m-1)\,k}\left(\frac{\sqrt{u}}{v^{m-1}} + (m-\tfrac{3}{2})\,b\int \frac{dx}{v^{m-1}\sqrt{u}}\right).$$

**120.** $$\int v^m u^{n-\frac{1}{2}}\,dx = \frac{1}{(2m+2n+1)\,b'}\left(2\,v^{m+1}u^{n-\frac{1}{2}} + (2n-1)\,k\int v^m u^{n-\frac{3}{2}}\,dx\right).$$

**121.** $$\int v^m u^{-(n+\frac{1}{2})}\,dx = \frac{1}{(2n-1)\,k}\left(2\,v^{m+1}u^{-(n-\frac{1}{2})} - (2m-2n+3)\,b'\int v^m u^{-(n-\frac{1}{2})}\,dx\right)$$

$$= \frac{2}{(2n-1)\,b}\left(-\,v^m u^{-(n-\frac{1}{2})} + mb'\int v^{m-1}u^{-(n-\frac{1}{2})}\,dx\right).$$

**122.** $$\int v^{-m}u^{(n-\frac{1}{2})}\,dx = \frac{-1}{(2m-2n-1)\,b'}\left(2\,u^{n-\frac{1}{2}}v^{-(m-1)} + (2n-1)\,k\int u^{n-\frac{3}{2}}v^{-m}\,dx\right)$$

$$= \frac{1}{(m-1)\,b'}\left(-\,u^{n-\frac{1}{2}}v^{-(m-1)} + (n-\tfrac{1}{2})\,b\int u^{n-\frac{3}{2}}v^{-(m-1)}\,dx\right).$$

**123.** $$\int v^{-m}u^{-(n+\frac{1}{2})}\,dx = \frac{1}{(2n-1)\,k}\left(2\,v^{-(m-1)}u^{-(n-\frac{1}{2})} + (2m+2n-3)\,b'\int v^{-m}u^{-(n-\frac{1}{2})}\,dx\right).$$

C. — Expressions Involving $\sqrt{x^2 \pm a^2}$ and $\sqrt{a^2 - x^2}$.

124. $\int \sqrt{x^2 \pm a^2}\, dx = \frac{1}{2}\left[x\sqrt{x^2 \pm a^2} \pm a^2 \log\left(x + \sqrt{x^2 \pm a^2}\right)\right]$.*

125. $\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{a}\right)$.

126 *a*. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left(x + \sqrt{x^2 + a^2}\right)$, or $\sinh^{-1}\frac{x}{a}$.*

126 *b*. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left(x + \sqrt{x^2 - a^2}\right)$, or $\cosh^{-1}\frac{x}{a}$.*

127. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a}$, or $-\cos^{-1}\frac{x}{a}$.

128. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\cos^{-1}\frac{a}{x}$.

129. $\int \frac{dx}{x\sqrt{a^2 \pm x^2}} = -\frac{1}{a}\log\left(\frac{a + \sqrt{a^2 \pm x^2}}{x}\right)$.*

130. $\int \frac{\sqrt{a^2 \pm x^2}}{x}\, dx = \sqrt{a^2 \pm x^2} - a \log\frac{a + \sqrt{a^2 \pm x^2}}{x}$.*

131. $\int \frac{\sqrt{x^2 - a^2}}{x}\, dx = \sqrt{x^2 - a^2} - a\cos^{-1}\frac{a}{x}$.

132. $\int \frac{x\, dx}{\sqrt{a^2 \pm x^2}} = \pm\sqrt{a^2 \pm x^2}$.

133. $\int \frac{x\, dx}{\sqrt{x^2 - a^2}} = \sqrt{x^2 - a^2}$.

* $\log\left(\frac{x + \sqrt{x^2 + a^2}}{a}\right) = \sinh^{-1}\left(\frac{x}{a}\right)$; $\log\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right) = \cosh^{-1}\left(\frac{x}{a}\right)$;

$\log\left(\frac{a + \sqrt{a^2 - x^2}}{x}\right) = \operatorname{sech}^{-1}\left(\frac{x}{a}\right)$; $\log\left(\frac{a + \sqrt{a^2 + x^2}}{x}\right) = \operatorname{csch}^{-1}\left(\frac{x}{a}\right)$;

$\log z = \sinh^{-1}\left(\frac{z^2 - 1}{2z}\right) = \cosh^{-1}\left(\frac{z^2 + 1}{2z}\right)$; $\tanh^{-1} z = -i \cdot \tan^{-1}(zi)$.

134. $\int x\sqrt{x^2 \pm a^2}\,dx = \frac{1}{3}\sqrt{(x^2 \pm a^2)^3}.$

135. $\int x\sqrt{a^2 - x^2}\,dx = -\frac{1}{3}\sqrt{(a^2 - x^2)^3}.$

136. $\int \sqrt{(x^2 \pm a^2)^3}\,dx$

$$= \tfrac{1}{4}\left[x\sqrt{(x^2 \pm a^2)^3} \pm \frac{3\,a^2 x}{2}\sqrt{x^2 \pm a^2} + \frac{3\,a^4}{2}\log\left(x + \sqrt{x^2 \pm a^2}\right)\right].*$$

137. $\int \sqrt{(a^2 - x^2)^3}\,dx$

$$= \tfrac{1}{4}\left[x\sqrt{(a^2 - x^2)^3} + \frac{3\,a^2 x}{2}\sqrt{a^2 - x^2} + \frac{3\,a^4}{2}\sin^{-1}\frac{x}{a}\right].$$

138. $\int \frac{dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{\pm x}{a^2\sqrt{x^2 \pm a^2}}.$

139. $\int \frac{dx}{\sqrt{(a^2 - x^2)^3}} = \frac{x}{a^2\sqrt{a^2 - x^2}}.$

140. $\int \frac{x\,dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{-1}{\sqrt{x^2 \pm a^2}}.$

141. $\int \frac{x\,dx}{\sqrt{(a^2 - x^2)^3}} = \frac{1}{\sqrt{a^2 - x^2}}.$

142. $\int x\sqrt{(x^2 \pm a^2)^3}\,dx = \frac{1}{5}\sqrt{(x^2 \pm a^2)^5}.$

143. $\int x\sqrt{(a^2 - x^2)^3}\,dx = -\frac{1}{5}\sqrt{(a^2 - x^2)^5}.$

144. $\int x^2\sqrt{x^2 \pm a^2}\,dx$

$$= \frac{x}{4}\sqrt{(x^2 \pm a^2)^3} \mp \frac{a^2}{8}x\sqrt{x^2 \pm a^2} - \frac{a^4}{8}\log\left(x + \sqrt{x^2 \pm a^2}\right).*$$

145. $\int x^2\sqrt{a^2 - x^2}\,dx$

$$= -\frac{x}{4}\sqrt{(a^2 - x^2)^3} + \frac{a^2}{8}\left(x\sqrt{a^2 - x^2} + a^2\sin^{-1}\frac{x}{a}\right).$$

* See Note on page 20.

146. $\int \frac{\sqrt{a^2 \pm x^2}\, dx}{x^3} = -\frac{\sqrt{a^2 \pm x^2}}{2x^2} \pm \frac{1}{2}\int \frac{dx}{x\sqrt{a^2 \pm x^2}}.$

147. $\int x^3 \sqrt{a^2 \pm x^2}\, dx = (\pm \tfrac{1}{5} x^2 - 1\tfrac{2}{5} a^2)\sqrt{(a^2 \pm x^2)^3}.$

148. $\int \frac{dx}{x^3\sqrt{a^2 \pm x^2}} = -\frac{\sqrt{a^2 \pm x^2}}{2a^2x^2} \mp \frac{1}{2a^2}\int \frac{dx}{x\sqrt{a^2 \pm x^2}}.$

149. $\int \frac{dx}{x^3\sqrt{x^2 - a^2}} = \frac{\sqrt{x^2 - a^2}}{2a^2x^2} + \frac{1}{2a^3}\cos^{-1}\frac{a}{x}.$

150. $\int \frac{x^2 dx}{\sqrt{x^2 \pm a^2}} = \frac{x}{2}\sqrt{x^2 \pm a^2} \mp \frac{a^2}{2}\log(x + \sqrt{x^2 \pm a^2}).$ *

151. $\int \frac{x^2 dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}.$

152. $\int \frac{dx}{x^2\sqrt{x^2 \pm a^2}} = \mp \frac{\sqrt{x^2 \pm a^2}}{a^2x}.$

153. $\int \frac{dx}{x^2\sqrt{a^2 - x^2}} = -\frac{\sqrt{a^2 - x^2}}{a^2x}.$

154. $\int \frac{\sqrt{x^2 \pm a^2}\, dx}{x^2} = -\frac{\sqrt{x^2 \pm a^2}}{x} + \log(x + \sqrt{x^2 \pm a^2}).$ *

155. $\int \frac{\sqrt{a^2 - x^2}}{x^2} dx = -\frac{\sqrt{a^2 - x^2}}{x} - \sin^{-1}\frac{x}{a}.$

156. $\int \frac{x^2 dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{-x}{\sqrt{x^2 \pm a^2}} + \log(x + \sqrt{x^2 \pm a^2}).$ *

157. $\int \frac{x^2 dx}{\sqrt{(a^2 - x^2)^3}} = \frac{x}{\sqrt{a^2 - x^2}} - \sin^{-1}\frac{x}{a}.$

* (See Note on page 20.)

158. $$\int \frac{f(x^2)\,dx}{\sqrt{a+cx^2}} = g\int f\left(\frac{au^2}{g^2-cu^2}\right)\frac{du}{(g^2-cu^2)},$$

where $u = \dfrac{gx}{\sqrt{a+cx^2}}$.

159. $$\int \frac{xf(x^2)\,dx}{\sqrt{a+cx^2}} = \frac{1}{c}\int f\left(\frac{u^2-a}{c}\right)du, \text{ where } u^2 = a+cx^2.$$

### D. — Expressions Involving $\sqrt{a+bx+cx^2}$.

Let $X = a+bx+cx^2$, $q = 4ac-b^2$, and $k = \dfrac{4c}{q}$. In order to rationalize the function $f(x, \sqrt{a+bx+cx^2})$ we may put $\sqrt{a+bx+cx^2} = \sqrt{\pm c}\sqrt{A+Bx\pm x^2}$, according as $c$ is positive or negative, and then substitute for $x$ a new variable $z$, such that

$$z = \sqrt{A+Bx+x^2} \pm x, \text{ if } c>0.$$

$$z = \frac{\sqrt{A+Bx-x^2}-\sqrt{A}}{x}, \text{ if } c<0 \text{ and } \frac{a}{-c}>0.$$

$$z = \sqrt{\frac{x-\beta}{a-x}}, \text{ where } \alpha \text{ and } \beta \text{ are the roots of the equation}$$

$$A+Bx-x^2 = 0, \text{ if } c<0 \text{ and } \frac{a}{-c}<0.$$

160. $$\int \frac{dx}{\sqrt{X}} = \frac{1}{\sqrt{c}}\log\left(\sqrt{X}+x\sqrt{c}+\frac{b}{2\sqrt{c}}\right),$$

$$\text{or } \frac{1}{\sqrt{c}}\sinh^{-1}\left(\frac{2cx+b}{\sqrt{q}}\right).$$

161. $$\int \frac{dx}{\sqrt{X}} = \frac{-1}{\sqrt{-c}}\sin^{-1}\left(\frac{2cx+b}{\sqrt{-g}}\right).$$

162. $\int \frac{dx}{X\sqrt{X}} = \frac{2(2cx+b)}{q\sqrt{X}}.$

163. $\int \frac{dx}{X^2\sqrt{X}} = \frac{2(2cx+b)}{3q\sqrt{X}}\left(\frac{1}{X}+2k\right).$

164. $\int \frac{dx}{X^n\sqrt{X}} = \frac{2(2cx+b)\sqrt{X}}{(2n-1)qX^n} + \frac{2k(n-1)}{2n-1}\int \frac{dx}{X^{n-1}\sqrt{X}}.$

165. $\int \sqrt{X}\,dx = \frac{(2cx+b)\sqrt{X}}{4c} + \frac{1}{2k}\int \frac{dx}{\sqrt{X}}.$

166. $\int X\sqrt{X}\,dx = \frac{(2cx+b)\sqrt{X}}{8c}\left(X+\frac{3}{2k}\right) + \frac{3}{8k^2}\int \frac{dx}{\sqrt{X}}.$

167. $\int X^2\sqrt{X}\,dx$

$$= \frac{(2cx+b)\sqrt{X}}{12c}\left(X^2+\frac{5X}{4k}+\frac{15}{8k^2}\right) + \frac{5}{16k^3}\int \frac{dx}{\sqrt{X}}.$$

168. $\int X^n\sqrt{X}\,dx = \frac{(2cx+b)X^n\sqrt{X}}{4(n+1)c} + \frac{2n+1}{2(n+1)k}\int \frac{X^n\,dx}{\sqrt{X}}.$

169. $\int \frac{x\,dx}{\sqrt{X}} = \frac{\sqrt{X}}{c} - \frac{b}{2c}\int \frac{dx}{\sqrt{X}}.$

170. $\int \frac{x\,dx}{X\sqrt{X}} = -\frac{2(bx+2a)}{q\sqrt{X}}.$

171. $\int \frac{x\,dx}{X^n\sqrt{X}} = -\frac{\sqrt{X}}{(2n-1)cX^n} - \frac{b}{2c}\int \frac{dx}{X^n\sqrt{X}}.$

172. $\int \frac{x^2\,dx}{\sqrt{X}} = \left(\frac{x}{2c}-\frac{3b}{4c^2}\right)\sqrt{X} + \frac{3b^2-4ac}{8c^2}\int \frac{dx}{\sqrt{X}}.$

173. $\int \frac{x^2\,dx}{X\sqrt{X}} = \frac{(2b^2-4ac)x+2ab}{cq\sqrt{X}} + \frac{1}{c}\int \frac{dx}{\sqrt{X}}.$

174. $$\int \frac{x^2\,dx}{X^n\sqrt{X}}$$

$$= \frac{(2b^2 - 4ac)x + 2ab}{(2n-1)cqX^{n-1}\sqrt{X}} + \frac{4ac + (2n-3)b^2}{(2n-1)cq}\int \frac{dx}{X^{n-1}\sqrt{X}}.$$

175. $$\int \frac{x^3\,dx}{\sqrt{X}}$$

$$= \left(\frac{x^2}{3c} - \frac{5bx}{12c^2} + \frac{5b^3}{8c^3} - \frac{2a}{3c^2}\right)\sqrt{X} + \left(\frac{3ab}{4c^2} - \frac{5b^2}{16c^3}\right)\int \frac{dx}{\sqrt{X}}.$$

176. $$\int x\sqrt{X}\,dx = \frac{X\sqrt{X}}{3c} - \frac{b}{2c}\int \sqrt{X}\,dx.$$

177. $$\int xX\sqrt{X}\,dx = \frac{X^2\sqrt{X}}{5c} - \frac{b}{2c}\int X\sqrt{X}\,dx.$$

178. $$\int \frac{xX^n\,dx}{\sqrt{X}} = \frac{X^n\sqrt{X}}{(2n+1)c} - \frac{b}{2c}\int \frac{X^n\,dx}{\sqrt{X}}.$$

179. $$\int x^2\sqrt{X}\,dx = \left(x - \frac{5b}{6c}\right)\frac{X\sqrt{X}}{4c} + \frac{5b^2 - 4ac}{16c^2}\int \sqrt{X}\,dx.$$

180. $$\int \frac{x^2X^n\,dx}{\sqrt{X}} = \frac{xX^n\sqrt{X}}{2(n+1)c} - \frac{(2n+3)b}{4(n+1)c}\int \frac{xX^n\,dx}{\sqrt{X}} - \frac{a}{2(n+1)c}\int \frac{X^n\,dx}{\sqrt{X}}.$$

181. $$\int x^3\sqrt{X}\,dx = \left(x^2 - \frac{7bx}{8c} + \frac{35b^2}{48c^2} - \frac{2a}{3c}\right)\frac{X\sqrt{X}}{5c} + \left(\frac{3ab}{8c^2} - \frac{7b^3}{32c^3}\right)\int \sqrt{X}\,dx.$$

182. $$\int \frac{dx}{x\sqrt{X}} = -\frac{1}{\sqrt{a}}\log\left(\frac{\sqrt{X} + \sqrt{a}}{x} + \frac{b}{2\sqrt{a}}\right), \text{ if } a > 0.$$

183. $\int \frac{dx}{x\sqrt{X}} = \frac{1}{\sqrt{-a}} \sin^{-1}\left(\frac{bx+2a}{x\sqrt{-q}}\right)$, or $\frac{-1}{\sqrt{a}} \sinh^{-1}\frac{2a+bx}{x\sqrt{q}}$.

184. $\int \frac{dx}{x\sqrt{X}} = -\frac{2\sqrt{X}}{bx}$, if $a = 0$.

185. $\int \frac{dx}{xX^n\sqrt{X}}$

$$= \frac{\sqrt{X}}{(2n-1)aX^n} + \frac{1}{a}\int \frac{dx}{xX^{n-1}\sqrt{X}} - \frac{b}{2a}\int \frac{dx}{X^n\sqrt{X}}.$$

186. $\int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{ax} - \frac{b}{2a}\int \frac{dx}{x\sqrt{X}}$.

187. $\int \frac{\sqrt{X}dx}{x} = \sqrt{X} + \frac{b}{2}\int \frac{dx}{\sqrt{X}} + a\int \frac{dx}{x\sqrt{X}}$.

188. $\int \frac{X^n dx}{x\sqrt{X}} = \frac{X^n}{(2n-1)\sqrt{X}} + a\int \frac{X^{n-1}dx}{x\sqrt{X}} + \frac{b}{2}\int \frac{X^{n-1}dx}{\sqrt{X}}$.

189. $\int \frac{\sqrt{X}dx}{x^2} = -\frac{\sqrt{X}}{x} + \frac{b}{2}\int \frac{dx}{x\sqrt{X}} + c\int \frac{dx}{\sqrt{X}}$.

190. $\int \frac{x^m dx}{X^n\sqrt{X}} = \frac{1}{c}\int \frac{x^{m-2}dx}{X^{n-1}\sqrt{X}} - \frac{b}{c}\int \frac{x^{m-1}dx}{X^n\sqrt{X}} - \frac{a}{c}\int \frac{x^{m-2}dx}{X^n\sqrt{X}}$.

191. $\int \frac{x^m X^n dx}{\sqrt{X}} = \frac{x^{m-1}X^n\sqrt{X}}{(2n+m)c} - \frac{(2n+2m-1)b}{2c(2n+m)}\int \frac{x^{m-1}X^n dx}{\sqrt{X}}$

$$- \frac{(m-1)a}{(2n+m)c}\int \frac{x^{m-2}X^n dx}{\sqrt{X}}.$$

192. $\int \frac{dx}{x^m X^n\sqrt{X}}$

$$= -\frac{\sqrt{X}}{(m-1)ax^{m-1}X^n} - \frac{(2n+2m-3)b}{2a(m-1)}\int \frac{dx}{x^{m-1}X^n\sqrt{X}}$$

$$- \frac{(2n+m-2)c}{(m-1)a}\int \frac{dx}{x^{m-2}X^n\sqrt{X}}.$$

193. $$\int \frac{X^n\,dx}{x^m\sqrt{X}} = -\frac{X^{n-1}\sqrt{X}}{(m-1)\,x^{m-1}} + \frac{(2n-1)\,b}{2\,(m-1)}\int \frac{X^{n-1}\,dx}{x^{m-1}\sqrt{X}} + \frac{(2n-1)\,c}{m-1}\int \frac{X^{n-1}\,dx}{x^{m-2}\sqrt{X}}.$$

194. $$\int f(x,\ \sqrt{(x-a)\,(x-b)})\,dx = 2\,(a-b)\int f\left\{\frac{bu^2-a}{u^2-1},\ \frac{u\,(b-a)}{u^2-1}\right\}\frac{u\,du}{(u^2-1)^2},$$

where $u^2(x-b) = x-a$.

E. — Expressions Involving Products of Powers of $(a'+b'x)$ and $\sqrt{a+bx+cx^2}$.

Let $X = a+bx+cx^2$, $v = a'+b'x$, $q = 4\,ac-b^2$,
$\beta = bb'-2\,a'c$, $k = ab'^2-a'bb'+ca'^2$, then

195. $$\int \frac{dx}{v\sqrt{X}} = \frac{1}{\sqrt{k}}\log\frac{2\,k+\beta v-2\,b'\sqrt{kX}}{v} = \frac{1}{\sqrt{-k}}\tan^{-1}\frac{2\,k+\beta v}{2b'\sqrt{-kX}} = \frac{1}{\sqrt{-k}}\sin^{-1}\frac{2\,k+\beta v}{b'v\sqrt{-q}},\ \text{if } k\neq 0.$$

196. $$\int \frac{dx}{v\sqrt{X}} = -\frac{2\,b'\sqrt{X}}{\beta v},\ \text{if } k=0:$$

thus, $$\int \frac{dx}{(x\pm 1)\sqrt{x^2-1}} = \pm\sqrt{\frac{x\mp 1}{x\pm 1}}.$$

197. $$\int \frac{dx}{v^2\sqrt{X}} = -\frac{b'\sqrt{X}}{kv} - \frac{\beta}{2\,k}\int \frac{dx}{v\sqrt{X}}.$$

198. $$\int \frac{dx}{v^2\sqrt{X}} = -\frac{2\,b'\sqrt{X}}{3\,\beta v^2} - \frac{2\,c}{3\,\beta}\int \frac{dx}{v\sqrt{X}},\ \text{if } k=0.$$

199. $\int \frac{dx}{vX\sqrt{X}} = \frac{1}{k}\left(\frac{b'}{\sqrt{X}} - \tfrac{1}{2}\beta\int \frac{dx}{X\sqrt{X}} + b'^2\int \frac{dx}{v\sqrt{X}}\right).$

200. $\int \frac{v\,dx}{X\sqrt{X}} = -\frac{2(2k+\beta v)}{b'q\sqrt{X}}.$

201. $\int \frac{v\,dx}{\sqrt{X}} = \frac{b'\sqrt{X}}{c} - \frac{\beta}{2c}\int \frac{dx}{\sqrt{X}}.$

202. $\int v\sqrt{X}\,dx = \frac{b'X\sqrt{X}}{3c} - \frac{\beta}{2c}\int \sqrt{X}\,dx.$

203. $\int \frac{v\,dx}{X^n\sqrt{X}} = -\frac{b'\sqrt{X}}{(2n-1)cX^n} - \frac{\beta}{2c}\int \frac{dx}{X^n\sqrt{X}}.$

204. $\int \frac{vX^n\,dx}{\sqrt{X}} = \frac{b'X^n\sqrt{X}}{(2n+1)c} - \frac{\beta}{2c}\int \frac{X^n dx}{\sqrt{X}}.$

205. $\int \frac{dx}{v^m\sqrt{X}} = -\frac{b'\sqrt{X}}{(m-1)kv^{m-1}} - \frac{(2m-3)\beta}{2(m-1)k}\int \frac{dx}{v^{m-1}\sqrt{X}}$
$$-\frac{(m-2)c}{(m-1)k}\int \frac{dx}{v^{m-2}\sqrt{X}}, \text{ if } k \neq 0.$$

206. $\int \frac{dx}{v^m\sqrt{X}} = -\frac{2b'\sqrt{X}}{(2m-1)\beta v^m}$
$$-\frac{2(m-1)c}{(2m-1)\beta}\int \frac{dx}{v^{m-1}\sqrt{X}}, \text{ if } k = 0.$$

207. $\int \frac{\sqrt{X}\,dx}{v^m} = -\frac{b'X\sqrt{X}}{(m-1)kv^{m-1}} - \frac{(2m-5)\beta}{2(m-1)k}\int \frac{\sqrt{X}\,dx}{v^{m-1}}$
$$-\frac{(m-4)c}{(m-1)k}\int \frac{\sqrt{X}\,dx}{v^{m-2}}$$
$$= \frac{1}{(m-1)b'^2}\left(-\frac{b'\sqrt{X}}{v^{m-1}} + \tfrac{1}{2}\beta\int \frac{dx}{v^{m-1}\sqrt{X}} + c\int \frac{dx}{v^{m-2}\sqrt{X}}\right)$$
$$= \frac{1}{(m-2)b'^2}\left(-\frac{b'\sqrt{X}}{v^{m-1}} - k\int \frac{dx}{v^m\sqrt{X}} - \tfrac{1}{2}\beta\int \frac{dx}{v^{m-1}\sqrt{X}}\right).$$

**208.** $$\int v^m\sqrt{X}\,dx = \frac{1}{(m+2)c}\Big(b'v^{m-1}X\sqrt{X} - (m+\tfrac{1}{2})\beta\int v^{m-1}\sqrt{X}\,dx - (m-1)k\int v^{m-2}\sqrt{X}\,dx\Big).$$

**209.** $$\int\frac{dx}{v^mX^n\sqrt{X}} = -\frac{1}{(m-1)k}\Big(\frac{b'\sqrt{X}}{v^{m-1}X^n} + (m+n-\tfrac{3}{2})\beta\int\frac{dx}{v^{m-1}X^n\sqrt{X}} + (m+2n-2)c\int\frac{dx}{v^{m-2}X^n\sqrt{X}}\Big),\ \text{if } k\neq 0.$$

**210.** $$\int\frac{dx}{v^mX^n\sqrt{X}} = \frac{-2}{(2m+2n-1)\beta}\Big(\frac{b'\sqrt{X}}{v^mX^n} + (m+2n-1)c\int\frac{dx}{v^{m-1}X^n\sqrt{X}}\Big),\ \text{if } k=0.$$

**211.** $$\int\frac{X^n\,dx}{v^m\sqrt{X}} = -\frac{1}{(m-1)k}\Big(\frac{b'X^n\sqrt{X}}{v^{m-1}} + (m-n-\tfrac{3}{2})\beta\int\frac{X^n\,dx}{v^{m-1}\sqrt{X}} + (m-2n-2)c\int\frac{X^n\,dx}{v^{m-2}\sqrt{X}}\Big)$$

$$= -\frac{1}{(m-2n)b'^2}\Big(\frac{b'X^{n-1}\sqrt{X}}{v^{m-1}} + (2n-1)k\int\frac{X^{n-1}dx}{v^m\sqrt{X}} + (n-\tfrac{1}{2})\beta\int\frac{X^{n-1}dx}{v^{m-1}\sqrt{X}}\Big)$$

$$= \frac{1}{(m-1)b'^2}\Big(-\frac{b'X^{n-1}\sqrt{X}}{v^{m-1}} + (n-\tfrac{1}{2})\beta\int\frac{X^{n-1}dx}{v^{m-1}\sqrt{X}} + (2n-1)c\int\frac{X^{n-1}dx}{v^{m-2}\sqrt{X}}\Big).$$

**212.** $$\int \frac{v^m X^n\,dx}{\sqrt{X}} = \frac{1}{(m+2n)c}\left(b' v^{m-1} X^n \sqrt{X} - (m+n-\tfrac{1}{2})\beta \int \frac{v^{m-1} X^n\,dx}{\sqrt{X}} - (m-1)k \int \frac{v^{m-2} X^n\,dx}{\sqrt{X}}\right).$$

**213.** $$\int \frac{v^m\,dx}{X^n\sqrt{X}} = \frac{1}{(m-2n)c}\left(\frac{b' v^{m-1}\sqrt{X}}{X^n} - (m-n-\tfrac{1}{2})\beta \int \frac{v^{m-1}\,dx}{X^n\sqrt{X}} - (m-1)k \int \frac{v^{m-2}\,dx}{X^n\sqrt{X}}\right).$$

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$$\frac{1}{(x+a)(x+b)\sqrt{X}} = \frac{1}{(b-a)(x+a)\sqrt{X}} + \frac{1}{(a-b)(x+b)\sqrt{X}}$$

$$\frac{1}{\sqrt{a+bx+cx^2} \pm \sqrt{a'+b'x+c'x^2}} = \frac{\sqrt{a+bx+cx^2} \mp \sqrt{a'+b'x+c'x^2}}{a-a'+(b-b')x+(c-c')x^2}.$$

$$\frac{\sqrt{X}}{(x+a)(x+b)} = \frac{\sqrt{X}}{(b-a)(x+a)} + \frac{\sqrt{X}}{(a-b)(x+b)}.$$

$$\frac{(x+a)\sqrt{X}}{x+b} = \sqrt{X} + \frac{(a-b)\sqrt{X}}{x+b}.$$

$\int \sqrt{\frac{ax^2+b}{a'x^2+b'}}\,dx$ is an elliptic integral.

$$\int \frac{x\sqrt{a+bx^2}}{\sqrt{a'+b'x^2}}\,dx = \frac{1}{b'\sqrt{b'}} \int \sqrt{ab' - a'b + by^2}\cdot dy,$$

where $y^2 = a' + b'x^2$.

## IV. MISCELLANEOUS ALGEBRAIC EXPRESSIONS.

214. $$\int \sqrt{2ax - x^2} \cdot dx = \frac{x-a}{2}\sqrt{2ax - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x-a}{a}.$$

215. $$\int \frac{dx}{\sqrt{2ax - x^2}} = \operatorname{versin}^{-1}\frac{x}{a} = \cos^{-1}\left(1 - \frac{x}{a}\right)$$
$$= 2\sin^{-1}\sqrt{\frac{x}{2a}}.$$

216. $$\int \frac{x^n\,dx}{\sqrt{2ax - x^2}} = -\frac{x^{n-1}\sqrt{2ax - x^2}}{n}$$
$$-\frac{a(1-2n)}{n}\int \frac{x^{n-1}dx}{\sqrt{2ax - x^2}}.$$

217. $$\int \frac{dx}{x^n\sqrt{2ax - x^2}} = \frac{\sqrt{2ax - x^2}}{a(1-2n)x^n}$$
$$+\frac{n-1}{(2n-1)a}\int \frac{dx}{x^{n-1}\sqrt{2ax - x^2}}.$$

218. $$\int x^n\sqrt{2ax - x^2}\cdot dx = -\frac{x^{n-1}\sqrt{(2ax - x^2)^3}}{n+2}$$
$$+\frac{(2n+1)a}{n+2}\int x^{n-1}\sqrt{2ax - x^2}\cdot dx$$

219. $$\int \frac{\sqrt{2ax - x^2}\cdot dx}{x^n} = \frac{\sqrt{(2ax - x^2)^3}}{(3-2n)ax^n}$$
$$+\frac{n-3}{(2n-3)a}\int \frac{\sqrt{2ax - x^2}\cdot dx}{x^{n-1}}.$$

220. $$\int \frac{dx}{x\sqrt{x^n - a^2}} = \frac{2}{an}\cos^{-1}\frac{a}{x^{\frac{n}{2}}}.$$

221. $$\int \frac{dx}{x\sqrt{x^n+a^2}} = \frac{1}{an}\log\frac{\sqrt{a^2+x^n}-a}{\sqrt{a^2+x^n}+a}.$$

222. $$\int \frac{x^{\frac{1}{2}}\,dx}{\sqrt{a^3-x^3}} = \tfrac{2}{3}\sin^{-1}\left(\frac{x}{a}\right)^{\frac{3}{2}}.$$

223. $$\int \frac{dx}{(a+bx^2)\sqrt{x}} = \frac{1}{b\delta^3\sqrt{2}}\left\{\log\left(\frac{x+\delta^2+\sqrt{2\,\delta^2 x}}{\sqrt{a+bx^2}}\right)\right.$$
$$\left. +\tan^{-1}\left(1+\frac{\sqrt{2\,x}}{\delta}\right)-\tan^{-1}\left(1-\frac{\sqrt{2\,x}}{\delta}\right)\right\}, \text{ where } b\delta^4 = a.$$

224. $$\int \frac{\sqrt{x}\cdot dx}{a+bx^2} = \frac{1}{b\delta\sqrt{2}}\left\{\tan^{-1}\left(1+\frac{\sqrt{2\,x}}{\delta}\right)-\tan^{-1}\left(1-\frac{\sqrt{2\,x}}{\delta}\right)\right.$$
$$\left. -\log\left(\frac{x+\delta^2+\sqrt{2\,\delta^2 x}}{\sqrt{a+bx^2}}\right)\right\}, \text{ where } b\delta^4 = a.$$

225. $$\int \frac{x^{\frac{3}{2}}\cdot dx}{a+bx^2} = \frac{2\sqrt{x}}{b} - \frac{a}{b}\int \frac{dx}{(a+bx^2)\sqrt{x}}.$$

226. $$\int \frac{dx}{(a+bx^2)^2\sqrt{x}} = \frac{\sqrt{x}}{2\,a\,(a+bx^2)} + \frac{3}{4\,a}\int \frac{dx}{(a+bx^2)\sqrt{x}}.$$

227. $$\int \frac{\sqrt{x}\cdot dx}{(a+bx^2)^2} = \frac{x^{\frac{3}{2}}}{2\,a\,(a+bx^2)} + \frac{1}{4\,a}\int \frac{\sqrt{x}\cdot dx}{(a+bx^2)}.$$

If $a_1$, $a_2$, $a_3$, etc., are the roots of the equation

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n = 0,$$

the integrand in the expression

$$\int \frac{(q_0x^m + q_1x^{m-1} + \cdots + q_n)\,dx}{(p_0x^n + p_1x^{n-1} + \cdots + p_n)\sqrt{a+bx+cx^2}},$$

where $m < n$, may be expressed as the sum of a number of partial fractions of the form $-\dfrac{A}{(x-a_k)^r\sqrt{a+bx+cx^2}}$, and these can be integrated by the aid of equations given above. Thus,

**228.** $$\int \frac{(px+q)\,dx}{(x-a')(x-b')\sqrt{a+bx+cx^2}}$$

$$= \frac{q+a'p}{a'-b'}\int \frac{dx}{(x-a')\sqrt{a+bx+cx^2}}$$

$$-\frac{q+b'p}{a'-b'}\int \frac{dx}{(x-b')\sqrt{a+bx+cx^2}}.$$

**229.** $$\int \frac{dx}{(a'+c'x^2)\sqrt{a+cx^2}}$$

$$= \frac{1}{a'}\sqrt{\frac{a'}{ac'-a'c}}\tan^{-1} x\sqrt{\frac{ac'-a'c}{a'(a+cx^2)}},$$

$$\text{or } \frac{1}{2\,a'}\sqrt{\frac{a'}{a'c-ac'}}\log\frac{\sqrt{a+cx^2}+x\sqrt{(a'c-ac')/a'}}{\sqrt{a+cx^2}-x\sqrt{(a'c-ac')/a'}}$$

**230.** $$\int \frac{x\,dx}{(a'+c'x^2)\sqrt{a+cx^2}}$$

$$= \frac{1}{c'}\sqrt{\frac{c'}{a'c-ac'}}\tan^{-1}\sqrt{\frac{c'(a+cx^2)}{a'c-ac'}},$$

$$\text{or } \frac{1}{2\,c'}\sqrt{\frac{c'}{ac'-a'c}}\log\frac{\sqrt{a+cx^2}-\sqrt{(ac'-a'c)/c'}}{\sqrt{a+cx^2}+\sqrt{(ac'-a'c)/c'}}.$$

**231.** $$\int f\left\{x,\ \sqrt[n]{\frac{a+bx}{a'+b'x}}\right\}dx$$

$$= n(a'b-ab')\int f\left(\frac{a-a'z^n}{b'z^n-b},\ z\right)\cdot\frac{z^{n-1}\,dz}{(b'z^n-b)^2},$$

where $z^n(a'+b'x)=a+bx$.

**232.** $\int f(x, \sqrt[n]{c + \sqrt[m]{a + bx}})\, dx$

$$= \frac{mn}{b} \int f\left\{ \frac{(z^n - c)^m - a}{b},\ z \right\} (z^n - c)^{m-1} z^{n-1}\, dz,$$

where $z^n = c + \sqrt[m]{a + bx}$.

**233.** $\int f\left\{ x, \left[\frac{a + bx}{a' + b'x}\right]^{\frac{m}{n}}, \left[\frac{a + bx}{a' + b'x}\right]^{\frac{p}{q}}, \cdots \right\} dx$

$$= s(a'b - ab') \int f\left\{ \frac{a'y^s - a}{b - b'y^s},\ y^{\frac{ms}{n}},\ y^{\frac{ps}{q}},\ \cdots \right\} \frac{y^{s-1}\, dy}{(b - b'y^s)^2},$$

where $y^s(a' + b'x) = a + bx$ and $s$ is the least common multiple of $n$, $q$, etc.

**234.** $\int f(x, \sqrt{a + bx + x^2})\, dx$

$$= 2 \int f\left( \frac{2\sqrt{a} \cdot z - b}{1 - z^2},\ \frac{z^2\sqrt{a} - bz + \sqrt{a}}{1 - z^2} \right) \cdot \frac{(z^2\sqrt{a} - bz + \sqrt{a})\, dz}{(1 - z^2)^2},$$

where $xz + \sqrt{a} = \sqrt{a + bx + x^2}$.

**235.** $\int f(x, \sqrt{a + bx + x^2})\, dx$

$$= \int f\left( \frac{u^2 - a}{b - 2u},\ \frac{u^2 - bu + a}{2u - b} \right) \frac{2(bu - a - u^2)\, du}{(b - 2u)^2},$$

where $u = \sqrt{a + bx + x^2} - x$.

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$$\int \frac{dx}{x^4 + a^4} = \frac{1}{4a^3\sqrt{2}} \left\{ \log\left( \frac{x^2 + ax\sqrt{2} + a^2}{x^2 - ax\sqrt{2} + a^2} \right) + 2\tan^{-1}\left( \frac{ax\sqrt{2}}{a^2 - x^2} \right) \right\}$$

$$\int \frac{dx}{x^4 - a^4} = \frac{1}{4a^3} \left\{ \log\left( \frac{x - a}{x + a} \right) - 2\tan^{-1}\left( \frac{x}{a} \right) \right\}.$$

## V. TRANSCENDENTAL FUNCTIONS.

**236.** $\int \sin x \cdot f(\cos x)\, dx = -\int f(\cos x)\, d\cos x.$

**237.** $\int \cos x \cdot f(\sin x)\, dx = \int f(\sin x)\, d\sin x.$

**238.** $\int \sin x \cdot f(\sin x,\ \cos x)\, dx = -\int f(\sqrt{1-z^2},\ z)\, dz,$
where $z = \cos x.$

**239.** $\int \frac{dx}{a + b\cos x} = \frac{1}{c(b-a)}\left\{\int \frac{dz}{z+c} - \int \frac{dz}{z-c}\right\},$
where $z = \tan \frac{1}{2}x,$ and $c^2 = (b+a)/(b-a).$ [See 651.]

**240.** $\int \frac{dx}{a \pm b\sin x} = \int \frac{2\,dz}{a \pm 2bz + az^2},$ where $z = \tan \frac{1}{2}x.$

**241.** $\int f(\sin x)\, dx = -\int f\left(\cos\left(\frac{\pi}{2} - x\right)\right) d\left(\frac{\pi}{2} - x\right).$

**242.** $\int f(\tan x)\, dx = -\int f \operatorname{ctn}\left(\frac{\pi}{2} - x\right) d\left(\frac{\pi}{2} - x\right).$

**243.** $\int f(\sec x)\, dx = -\int f \csc\left(\frac{\pi}{2} - x\right) d\left(\frac{\pi}{2} - x\right).$

**244.** $\int \frac{\sin x \cdot f(\sin^2 x)\, dx}{\sqrt{1 - k^2\sin^2 x}} = \int \frac{f(z)\, dz}{2\sqrt{(1-z)(1-k^2z)}},$
where $z = \sin^2 x.$

**245.** $\int \frac{\cos x \cdot f(\cos^2 x)\, dx}{\sqrt{1 - k^2\sin^2 x}} = \int \frac{f(1-z)\, dz}{2\sqrt{z(1-k^2z)}},$ where $z = \sin^2 x.$

**246.** $\int \frac{\tan x \cdot f(\tan^2 x)\,dx}{\sqrt{1-k^2\sin^2 x}} = \int f\left(\frac{z}{1-z}\right)\frac{dz}{2(1-z)\sqrt{1-k^2 z}},$

where $z = \sin^2 x$.

**247.** $\int f(ax+b)\,dx = \frac{1}{a}\int f(ax+b)\,d(ax+b).$

**248.** $\int \sec^{n+2} x \cdot f(\tan x)\,dx = \int (1+z^2)^{\frac{n}{2}} f(z)\,dz;\ z = \tan x.$

**249.** $\int f(\sin x,\ \cos x)\,dx$

$$= -\int f\left(\cos\left(\frac{\pi}{2}-x\right),\ \sin\left(\frac{\pi}{2}-x\right)\right)d\left(\frac{\pi}{2}-x\right)$$

**250.** $\int f(x)\cdot \sin^{-1} x \cdot dx = \sin^{-1} x \cdot \phi(x) - \int \frac{\phi(x)\,dx}{\sqrt{1-x^2}},\ dx,$

where $\phi(x) = \int f(x)\,dx.$

**251.** $\int f(x)\cdot \cos^{-1} x\,dx = \cos^{-1} x \cdot \phi(x) + \int \frac{\phi(x)\,dx}{\sqrt{1-x^2}}.$

**252.** $\int f(x)\cdot \tan^{-1} x\,dx = \tan^{-1} x \cdot \phi(x) - \int \frac{\phi(x)\,dx}{1+x^2}.$

**253.** $\int f(x)\cdot \operatorname{ctn}^{-1} x\,dx = \operatorname{ctn}^{-1} x \cdot \phi(x) + \int \frac{\phi(x)\,dx}{1+x^2}.$

**254.** $\int f(x,\ \cos x)\,dx = -\int f\left(\frac{\pi}{2}-z,\ \sin z\right)dz,$

where $z = \frac{\pi}{2} - x.$

**255.** $\int \frac{\sin x \cdot f(\cos x)\,dx}{a+b\cos x} = -\frac{1}{b}\int f\left(\frac{z-a}{b}\right)\frac{dz}{z}.$

where $z = a + b\cos x.$

**256.** $\int f(x,\ \log x)\,dx = \int f(e^z,\ z)\,e^z\,dz,$ where $z = \log x.$

**257.** $\int \frac{f(\log x)\,dx}{x} = \int f(z)\,dz,$ where $z = \log x.$

**258.** $\int x^m f(\log x)\,dx = \int e^{(m+1)z} f(z)\,dz.$

**259.** $\int f(\sin x,\ \cos x,\ \tan x,\ \operatorname{ctn} x,\ \sec x,\ \csc x)\,dx$

$$= \int f\left(\frac{2z}{1+z^2},\ \frac{1-z^2}{1+z^2},\ \frac{2z}{1-z^2},\ \frac{1-z^2}{2z},\ \frac{1+z^2}{1-z^2},\ \frac{1+z^2}{2z}\right)$$

$\frac{2\,dz}{1+z^2}$, where $z = \tan\frac{x}{2}$;

$$= \int f\left(z,\ \sqrt{1-z^2},\ \frac{z}{\sqrt{1-z^2}},\ \frac{\sqrt{1-z^2}}{z},\ \frac{1}{\sqrt{1-z^2}},\ \frac{1}{z}\right)$$

$\frac{dz}{\sqrt{1-z^2}}$, where $z = \sin x$;

$$= \int f\left(\frac{z}{\sqrt{1+z^2}},\ \frac{1}{\sqrt{1+z^2}},\ z,\ \frac{1}{z},\ \sqrt{1+z^2},\ \frac{\sqrt{1+z^2}}{z}\right)$$

$\frac{dz}{1+z^2}$, where $z = \tan x$;

$$= \int f\left(\sqrt{z},\ \sqrt{1-z},\ \sqrt{\frac{z}{1-z}},\ \sqrt{\frac{1-z}{z}},\ \frac{1}{\sqrt{1-z}},\ \frac{1}{\sqrt{z}}\right)$$

$\frac{dz}{2\sqrt{z(1-z)}}$, where $z = \sin^2 x$;

$$= \int f\left(\sqrt{\frac{z}{1+z}},\ \frac{1}{\sqrt{1+z}},\ \sqrt{z},\ \frac{1}{\sqrt{z}},\ \sqrt{1+z},\ \sqrt{\frac{1+z}{z}}\right)$$

$\frac{dz}{2(1+z)\sqrt{z}}$, where $z = \tan^2 x.$

260. $\int \sin x\,dx = -\cos x.$ [See 247.]

261. $\int \sin^2 x\,dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x = \frac{1}{2}x - \frac{1}{4}\sin 2x.$

262. $\int \sin^3 x\,dx = -\frac{1}{3}\cos x\,(\sin^2 x + 2).$

263. $\int \sin^n x\,dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx.$

264. $\int \cos x\,dx = \sin x.$ [See 247.]

265. $\int \cos^2 x\,dx = \frac{1}{2}\sin x \cos x + \frac{1}{2}x = \frac{1}{2}x + \frac{1}{4}\sin 2x.$

266. $\int \cos^3 x\,dx = \frac{1}{3}\sin x\,(\cos^2 x + 2).$

267. $\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx.$

268. $\int \sin x \cos x\,dx = \frac{1}{2}\sin^2 x.$

269. $\int \sin^2 x \cos^2 x\,dx = -\frac{1}{8}\left(\frac{1}{4}\sin 4x - x\right).$

270. $\int \sin x \cos^m x\,dx = -\frac{\cos^{m+1} x}{m+1}.$

271. $\int \sin^m x \cos x\,dx = \frac{\sin^{m+1} x}{m+1}.$

272. $\int \cos^m x \sin^n x\,dx = \frac{\cos^{m-1} x \sin^{n+1} x}{m+n} + \frac{m-1}{m+n}\int \cos^{m-2} x \sin^n x\,dx.$

273. $\int \cos^m x \sin^n x\,dx = -\frac{\sin^{n-1} x \cos^{m+1} x}{m+n} + \frac{n-1}{m+n}\int \cos^m x \sin^{n-2} x\,dx.$

274. $$\int \frac{\sin^n x\,dx}{\cos^m x} = \frac{1}{n-m}\left(-\frac{\sin^{n-1}x}{\cos^{m-1}x} + (n-1)\int \frac{\sin^{n-2}x\,dx}{\cos^m x}\right)$$

$$= \frac{1}{m-1}\left(\frac{\sin^{n+1}x}{\cos^{m-1}x} - (n-m+2)\int \frac{\sin^n x\,dx}{\cos^{m-2}x}\right)$$

$$= \frac{1}{m-1}\left(\frac{\sin^{n-1}x}{\cos^{m-1}x} - (n-1)\int \frac{\sin^{n-2}x\,dx}{\cos^{m-2}x}\right).$$

275. $$\int \frac{\cos^m x\,dx}{\sin^n x} = -\frac{\cos^{m+1}x}{(n-1)\sin^{n-1}x} - \frac{m-n+2}{n-1}\int \frac{\cos^m x\,dx}{\sin^{n-2}x}$$

$$= \frac{\cos^{m-1}x}{(m-n)\sin^{n-1}x} + \frac{m-1}{m-n}\int \frac{\cos^{m-2}x\,dx}{\sin^n x}$$

$$= -\frac{1}{n-1}\frac{\cos^{m-1}x}{\sin^{n-1}x} - \frac{m-1}{n-1}\int \frac{\cos^{m-2}x\,dx}{\sin^{n-2}x}.$$

276. $$\int \frac{\sin^m x\,dx}{\cos^n x} = -\int \frac{\cos^m\left(\frac{\pi}{2}-x\right)d\left(\frac{\pi}{2}-x\right)}{\sin^n\left(\frac{\pi}{2}-x\right)}.$$

277. $$\int \frac{dx}{\sin x\cos x} = \log\tan x.$$

278. $$\int \frac{dx}{\cos x\sin^2 x} = \log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) - \csc x.$$

279. $$\int \frac{dx}{\sin^m x\cos^n x}$$

$$= \frac{1}{n-1}\cdot\frac{1}{\sin^{m-1}x\cdot\cos^{n-1}x} + \frac{m+n-2}{n-1}\int \frac{dx}{\sin^m x\cdot\cos^{n-2}x}$$

$$= -\frac{1}{m-1}\cdot\frac{1}{\sin^{m-1}x\cdot\cos^{n-1}x} + \frac{m+n-2}{m-1}\int \frac{dx}{\sin^{m-2}x\cdot\cos^n x}.$$

280. $$\int \frac{dx}{\sin^m x} = -\frac{1}{m-1}\cdot\frac{\cos x}{\sin^{m-1}x} + \frac{m-2}{m-1}\int \frac{dx}{\sin^{m-2}x}.$$

281. $\int \frac{dx}{\cos^n x} = \frac{1}{n-1} \cdot \frac{\sin x}{\cos^{n-1} x} + \frac{n-2}{n-1} \int \frac{dx}{\cos^{n-2} x}.$

282. $\int \tan x \, dx = -\log \cos x.$ [See 247.]

283. $\int \tan^2 x \, dx = \tan x - x.$

284. $\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx.$

285. $\int \operatorname{ctn} x \, dx = \log \sin x.$ [See 247.]

286. $\int \operatorname{ctn}^2 x \, dx = -\operatorname{ctn} x - x.$

287. $\int \operatorname{ctn}^n x \, dx = -\frac{\operatorname{ctn}^{n-1} x}{n-1} - \int \operatorname{ctn}^{n-2} x \, dx.$

288. $\int \sec x \, dx = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) = \frac{1}{2} \log \frac{1+\sin x}{1-\sin x}.$

289. $\int \sec^2 x \, dx = \tan x.$

290. $\int \sec^n x \, dx = \int \frac{dx}{\cos^n x} = \frac{\sin x}{(n-1)\cos^{n-1} x} + \frac{n-2}{n-1} \int \frac{dx}{\cos^{n-2} x}$

$$= \frac{\sin x}{(n-1)\cos^{n-1} x} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx.$$

291. $\int \csc x \, dx = \log \tan \tfrac{1}{2} x.$

292. $\int \csc^2 x \, dx = -\operatorname{ctn} x.$

**293.** $\int \csc^n x\,dx = -\frac{\cos x}{(n-1)\sin^{n-1}x} + \frac{n-2}{n-1}\int \csc^{n-2}x\,dx.$

**294.** $\int \frac{dx}{1+\sin x} = -\tan\left(\tfrac{1}{4}\pi - \tfrac{1}{2}x\right).$

**295.** $\int \frac{dx}{1-\sin x} = \operatorname{ctn}\left(\tfrac{1}{4}\pi - \tfrac{1}{2}x\right) = \tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}x\right).$

**296.** $\int \frac{dx}{1+\cos x} = \tan \tfrac{1}{2}x, \quad \text{or} \quad \csc x - \operatorname{ctn} x.$

**297.** $\int \frac{dx}{1-\cos x} = -\operatorname{ctn} \tfrac{1}{2}x, \quad \text{or} \quad -\operatorname{ctn} x - \csc x.$

**298.** $\int \frac{dx}{a+b\sin x} = \frac{2}{\sqrt{a^2-b^2}} \tan^{-1}\frac{a\tan\frac{1}{2}x+b}{\sqrt{a^2-b^2}},$

or $\frac{1}{\sqrt{b^2-a^2}} \log \frac{a\tan\frac{1}{2}x+b-\sqrt{b^2-a^2}}{a\tan\frac{1}{2}x+b+\sqrt{b^2-a^2}},$

or $\frac{-2}{\sqrt{b^2-a^2}} \tanh^{-1}\frac{a\tan\frac{1}{2}x+b}{\sqrt{b^2-a^2}},$

or $\frac{-2}{\sqrt{b^2-a^2}} \operatorname{ctnh}^{-1}\frac{a\tan\frac{1}{2}x+b}{\sqrt{b^2-a^2}}.$

**299.** $\int \frac{dx}{a+b\sin x} = \frac{1}{b\cos\alpha} \log \frac{\sin\frac{1}{2}(x+\alpha)}{\cos\frac{1}{2}(x-\alpha)},$

$a = b\sin\alpha, \quad \sqrt{b^2-a^2} = b\cos\alpha, \quad -\pi < x < \pi.$

**300.** $\int \frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}} \tan^{-1}\frac{\sqrt{a^2-b^2}\tan\frac{1}{2}x}{a+b},$

or $\frac{1}{\sqrt{b^2-a^2}} \log \frac{\sqrt{b^2-a^2}\tan\frac{1}{2}x+a+b}{\sqrt{b^2-a^2}\tan\frac{1}{2}x-a-b},$

or $\frac{2}{\sqrt{b^2-a^2}} \tanh^{-1}\frac{\sqrt{b^2-a^2}\tan\frac{1}{2}x}{a+b},$

or $\frac{2}{\sqrt{b^2-a^2}} \operatorname{ctnh}^{-1}\frac{\sqrt{b^2-a^2}\tan\frac{1}{2}x}{a+b}.$

301. $\int \frac{dx}{a + b \tan x} = \frac{1}{a^2 + b^2}[b \log (a \cos x + b \sin x) + ax]$.

302. $\int \frac{dx}{\sin x + \cos x} = \frac{1}{\sqrt{2}} \log \tan (\frac{1}{2} x + \frac{1}{8} \pi)$.

303. $\int \frac{\sin x \, dx}{a + b \cos x} = -\frac{1}{b} \log (a + b \cos x)$.

304. $\int \frac{(a' + b' \cos x) \, dx}{a + b \cos x} = \frac{b'x}{b} + \frac{a'b - ab'}{b} \int \frac{dx}{a + b \cos x}$.

305. $\int \frac{(a' + b' \cos x) \, dx}{(a + b \cos x)^2} = \frac{ab' - a'b}{a^2 - b^2} \frac{\sin x}{a + b \cos x}$

$$+ \frac{aa' - bb'}{a^2 - b^2} \int \frac{dx}{a + b \cos x}.$$ [See 241.]

306. $\int \frac{(a' + b' \cos x) \, dx}{(a + b \cos x)^n} = \frac{1}{(n-1)(a^2 - b^2)} \left[ \frac{(ab' - a'b) \sin x}{(a + b \cos x)^{n-1}} \right.$

$$\left. + \int \frac{[(aa' - bb')(n-1) + (n-2)(ab' - a'b) \cos x] \, dx}{(a + b \cos x)^{n-1}} \right].$$

307. $\int \frac{(a' + b' \cos x) \, dx}{(1 + \cos x)^n} = \frac{(a' - b') \tan \frac{1}{2} x}{(2n-1)(1 + \cos x)^{n-1}}$

$$+ \frac{n(a' + b') - a'}{2n - 1} \int \frac{dx}{(1 + \cos x)^{n-1}}.$$

308. $\int \frac{dx}{(a + b \cos x)^n} = \frac{1}{(n-1)(a^2 - b^2)} \left[ \frac{-b \sin x}{(a + b \cos x)^{n-1}} \right.$

$$\left. + (2n - 3) a \int \frac{dx}{(a + b \cos x)^{n-1}} - (n-2) \int \frac{dx}{(a + b \cos x)^{n-2}} \right].$$

**309.** $$\int \frac{dx}{(1+\cos x)^n} = \frac{\tan \frac{1}{2}x}{(2n-1)(1+\cos x)^{n-1}} + \frac{n-1}{2n-1}\int \frac{dx}{(1+\cos x)^{n-1}}.$$ [See 241.]

**310.** $$\int \frac{(a'+b'\cos x)\,dx}{\sin x(a+b\cos x)} = \frac{a'b-ab'}{a^2-b^2}\log(a+b\cos x) + \frac{a'+b'}{a+b}\log \sin \tfrac{1}{2}x - \frac{a'-b'}{a-b}\log \cos \tfrac{1}{2}x.$$

**311.** $$\int \frac{(a'+b'\cos x)\,dx}{\cos x(a+b\cos x)} = \frac{a'}{a}\log \tan \tfrac{1}{2}(\tfrac{1}{2}\pi + x) + \frac{(ab'-a'b)}{a}\int \frac{dx}{a+b\cos x}.$$

**312.** $$\int \frac{(a'+b'\cos x)\,dx}{\sin x(1\pm\cos x)} = \pm\frac{\frac{1}{2}(a'\mp b')}{1\pm\cos x} + \tfrac{1}{2}(a'\pm b')\log \tan \tfrac{1}{2}x.$$

**313.** $$\int \frac{dx}{(1-\cos x)^n} = \frac{-\operatorname{ctn}\frac{1}{2}x}{(2n-1)(1-\cos x)^{n-1}} + \frac{n-1}{2n-1}\int \frac{dx}{(1-\cos x)^{n-1}}.$$ [See 241.]

**314.** $$\int \frac{dx}{a+b\sin^2 x} = \frac{1}{\sqrt{a^2+ab}}\tan^{-1}\frac{\sqrt{a^2+ab}\,\tan x}{a},$$

or $$\frac{1}{2\sqrt{-a^2-ab}}\log\frac{\sqrt{-a^2-ab}\,\tan x + a}{\sqrt{-a^2-ab}\,\tan x - a},$$

or $$\frac{1}{\sqrt{-a^2-ab}}\tanh^{-1}\frac{\sqrt{-a^2-ab}\,\tan x}{a},$$

or $$\frac{1}{\sqrt{-a^2-ab}}\operatorname{ctnh}^{-1}\frac{\sqrt{-a^2-ab}\,\tan x}{a}.$$

**315.** $$\int \frac{dx}{a+b\cos^2 x} = \frac{1}{\sqrt{a^2+ab}} \tan^{-1} \frac{\sqrt{a^2+ab}\tan x}{a+b},$$

$$\text{or } \frac{1}{2\sqrt{-a^2-ab}} \log \frac{\sqrt{-a^2-ab}\tan x + a + b}{\sqrt{-a^2-ab}\tan x - a - b},$$

$$\text{or } \frac{1}{\sqrt{-a^2-ab}} \tanh^{-1} \frac{\sqrt{-a^2-ab}\tan x}{a+b},$$

$$\text{or } \frac{1}{\sqrt{-a^2-ab}} \operatorname{ctnh}^{-1} \frac{\sqrt{-a^2-ab}\tan x}{a+b}.$$

**316.** $$\int \frac{dx}{a\cos^2 x + b\sin^2 x} = \frac{1}{\sqrt{ab}} \tan^{-1} \frac{\sqrt{ab}\tan x}{a},$$

$$\text{or } \frac{1}{2\sqrt{-ab}} \log \frac{\sqrt{-ab}\tan x + a}{\sqrt{-ab}\tan x - a},$$

$$\text{or } \frac{1}{\sqrt{-ab}} \tanh^{-1} \frac{\sqrt{-ab}\tan x}{a},$$

$$\text{or } \frac{1}{\sqrt{-ab}} \operatorname{ctnh}^{-1} \frac{\sqrt{-ab}\tan x}{a}.$$

**317.** $$\int \frac{\sin x \cos x\, dx}{a\cos^2 x + b\sin^2 x} = \frac{1}{2(b-a)} \log(a\cos^2 x + b\sin^2 x).$$

**318.** $$\int \frac{dx}{(a + b\cos x + c\sin x)^n} = \int \frac{d(x-\alpha)}{[a + r\cos(x-\alpha)]^n},$$

where $b = r\cos\alpha$ and $c = r\sin\alpha$.

**319.** $$\int \frac{dx}{a + b\cos x + c\sin x}$$

$$= \frac{2}{\sqrt{a^2-b^2-c^2}} \tan^{-1} \frac{(a-b)\tan\frac{1}{2}x + c}{\sqrt{a^2-b^2-c^2}},$$

$$\text{or } \frac{1}{\sqrt{b^2+c^2-a^2}} \log \frac{(a-b)\tan\frac{1}{2}x + c - \sqrt{b^2+c^2-a^2}}{(a-b)\tan\frac{1}{2}x + c + \sqrt{b^2+c^2-a^2}},$$

$$\text{or } \frac{-2}{\sqrt{b^2+c^2-a^2}} \tanh^{-1} \frac{(a-b)\tan\frac{1}{2}x + c}{\sqrt{b^2+c^2-a^2}},$$

$$\text{or } \frac{-2}{\sqrt{b^2+c^2-a^2}} \operatorname{ctnh}^{-1} \frac{(a-b)\tan\frac{1}{2}x + c}{\sqrt{b^2+c^2-a^2}}.$$

**320.** $\int \frac{dx}{a(1+\cos x)+c\sin x} = \frac{1}{c}\log(a + c\tan \tfrac{1}{2}x).$

**321.** $\int \frac{dx}{(a[1+\cos x]+c\sin x)^2}$

$$= \frac{1}{c^3}\left[\frac{c(a\sin x - c\cos x)}{a(1+\cos x)+c\sin x} - a\log(a + c\tan \tfrac{1}{2}x)\right]$$

**322.** $\int \frac{(x+\sin x)\,dx}{1+\cos x} = x\tan \tfrac{1}{2}x.$

**323.** $\int \cos x\sqrt{1-k^2\sin^2 x}\,dx$

$$= \tfrac{1}{2}\sin x\sqrt{1-k^2\sin^2 x} + \frac{1}{2k}\sin^{-1}(k\sin x).$$

**324.** $\int \sin x\sqrt{1-k^2\sin^2 x}\,dx$

$$= -\tfrac{1}{2}\cos x\sqrt{1-k^2\sin^2 x} - \frac{1-k^2}{2k}\log(k\cos x + \sqrt{1-k^2\sin^2 x}).$$

**325.** $\int \sin x(1-k^2\sin^2 x)^{\frac{3}{2}}\,dx = -\tfrac{1}{4}\cos x(1-k^2\sin^2 x)^{\frac{3}{2}}$

$$+ \tfrac{3}{4}(1-k^2)\int \sin x\sqrt{1-k^2\sin^2 x}\,dx.$$

**326.** $\int \frac{\cos x\,dx}{\sqrt{1-k^2\sin^2 x}} = \frac{1}{k}\sin^{-1}(k\sin x),$

$$\text{or } \frac{1}{b}\log(b\sin x + \sqrt{1+b^2\sin^2 x}), \text{ where } b^2 = -k^2.$$

**327.** $\int \frac{\sin x\,dx}{\sqrt{1-k^2\sin^2 x}} = -\frac{1}{k}\log(k\cos x + \sqrt{1-k^2\sin^2 x}),$

$$\text{or } -\frac{1}{b}\sin^{-1}\frac{b\cos x}{\sqrt{1+b^2}}, \text{ where } b^2 = -k^2$$

**328.** $\int \frac{\tan x\,dx}{\sqrt{1-k^2\sin^2 x}}$

$$= \frac{1}{2\sqrt{1-k^2}}\log\left(\frac{\sqrt{1-k^2\sin^2 x}+\sqrt{1-k^2}}{\sqrt{1-k^2\sin^2 x}-\sqrt{1-k^2}}\right).$$

329. $\int \frac{x\,dx}{1+\sin x} = -x\tan\tfrac{1}{2}(\tfrac{1}{2}\pi - x) + 2\log\cos\tfrac{1}{2}(\tfrac{1}{2}\pi - x).$

330. $\int \frac{x\,dx}{1-\sin x} = x\operatorname{ctn}\tfrac{1}{2}(\tfrac{1}{2}\pi - x) + 2\log\sin\tfrac{1}{2}(\tfrac{1}{2}\pi - x).$

331. $\int \frac{x\,dx}{1+\cos x} = x\tan\tfrac{1}{2}x + 2\log\cos\tfrac{1}{2}x.$

332. $\int \frac{x\,dx}{1-\cos x} = -x\operatorname{ctn}\tfrac{1}{2}x + 2\log\sin\tfrac{1}{2}x.$

333. $\int \frac{\tan x\,dx}{\sqrt{a+b\tan^2 x}} = \frac{1}{\sqrt{b-a}}\cos^{-1}\left(\frac{\sqrt{b-a}}{\sqrt{b}}\cdot\cos x\right).$

334. $\int \frac{dx}{a+b\tan^2 x} = \frac{1}{a-b}\left[x - \sqrt{\frac{b}{a}}\cdot\tan^{-1}\left(\sqrt{\frac{b}{a}}\cdot\tan x\right)\right].$

335. $\int \frac{\tan x\,dx}{a+b\tan x}$

$$= \frac{1}{a^2+b^2}\left\{bx - a\log(a+b\tan x) + a\log\sec x\right\}$$

336. $\int x\sin x\,dx = \sin x - x\cos x.$

337. $\int x^2\sin x\,dx = 2x\sin x - (x^2-2)\cos x.$

338. $\int x^3\sin x\,dx = (3x^2-6)\sin x - (x^3-6x)\cos x.$

339. $\int x^m\sin x\,dx = -x^m\cos x + m\int x^{m-1}\cos x\,dx.$

340. $\int x\cos x\,dx = \cos x + x\sin x.$

341. $\int x^2\cos x\,dx = 2x\cos x + (x^2-2)\sin x.$

342. $\int x^3\cos x\,dx = (3x^2-6)\cos x + (x^3-6x)\sin x.$

**343.** $\int x^m \cos x\,dx = x^m \sin x - m\int x^{m-1}\sin x\,dx.$

**344.** $\int \frac{\sin x}{x^m}\,dx = -\frac{1}{m-1}\cdot\frac{\sin x}{x^{m-1}} + \frac{1}{m-1}\int \frac{\cos x}{x^{m-1}}\,dx.$

**345.** $\int \frac{\cos x}{x^m}\,dx = -\frac{1}{m-1}\cdot\frac{\cos x}{x^{m-1}} - \frac{1}{m-1}\int \frac{\sin x}{x^{m-1}}\,dx.$

**346.** $\int \frac{\sin x}{x}\,dx = x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \frac{x^7}{7\cdot 7!} + \frac{x^9}{9\cdot 9!}\cdots.$

**347.** $\int \frac{\cos x}{x}\,dx = \log x - \frac{x^2}{2\cdot 2!} + \frac{x^4}{4\cdot 4!} - \frac{x^6}{6\cdot 6!} + \frac{x^8}{8\cdot 8!}\cdots.$

**348.** $\int \frac{x\,dx}{\sin x} = x + \frac{x^3}{3\cdot 3!} + \frac{7\,x^5}{3\cdot 5\cdot 5!} + \frac{31\,x^7}{3\cdot 7\cdot 7!} + \frac{127\,x^9}{3\cdot 5\cdot 9!} + \cdots$

**349.** $\int \frac{x\,dx}{\cos x} = \frac{x^2}{2} + \frac{x^4}{4\cdot 2!} + \frac{5\,x^6}{6\cdot 4!} + \frac{61\,x^8}{8\cdot 6!} + \frac{1385\,x^{10}}{10\cdot 8!} + \cdots.$

**350.** $\int \frac{x\,dx}{\sin^2 x} = -x\operatorname{ctn} x + \log\sin x.$

**351.** $\int \frac{x\,dx}{\cos^2 x} = x\tan x + \log\cos x.$

**352.** $n^2\int x^m \sin^n x\,dx$

$= x^{m-1}\sin^{n-1}x\,(m\sin x - nx\cos x)$

$+ n(n-1)\int x^m \sin^{n-2}x\,dx - m(m-1)\int x^{m-2}\sin^n x\,dx.$

**353.** $n^2\int x^m \cos^n x\,dx$

$= x^{m-1}\cos^{n-1}x\,(m\cos x + nx\sin x)$

$+ n(n-1)\int x^m \cos^{n-2}x\,dx - m(m-1)\int x^{m-2}\cos^n x\,dx$

**354.** $$\int \frac{x^m\,dx}{\sin^n x}$$

$$= \frac{1}{(n-1)(n-2)}\left[ - \frac{x^{m-1}(m \sin x + (n-2)\, x \cos x)}{\sin^{n-1} x} \right.$$

$$\left. + (n-2)^2 \int \frac{x^m\,dx}{\sin^{n-2} x} + m(m-1) \int \frac{x^{m-2}\,dx}{\sin^{n-2} x} \right].$$

**355.** $$\int \frac{x^m\,dx}{\cos^n x}$$

$$= \frac{1}{(n-1)(n-2)}\left[ - \frac{x^{m-1}(m \cos x - (n-2)\, x \sin x)}{\cos^{n-1} x} \right.$$

$$\left. + (n-2)^2 \int \frac{x^m\,dx}{\cos^{n-2} x} + m(m-1) \int \frac{x^{m-2}\,dx}{\cos^{n-2} x} \right].$$

**356.** $$\int \frac{\sin^n x\,dx}{x^m}$$

$$= \frac{1}{(m-1)(m-2)}\left[ - \frac{\sin^{n-1} x\,((m-2)\sin x + nx \cos x)}{x^{m-1}} \right.$$

$$\left. - n^2 \int \frac{\sin^n x\,dx}{x^{m-2}} + n(n-1) \int \frac{\sin^{n-2} x\,dx}{x^{m-2}} \right].$$

**357.** $$\int \frac{\cos^n x\,dx}{x^m}$$

$$= \frac{1}{(m-1)(m-2)}\left[ \frac{\cos^{n-1} x\,(nx \sin x - (m-2)\cos x)}{x^{m-1}} \right.$$

$$\left. - n^2 \int \frac{\cos^n x\,dx}{x^{m-2}} + n(n-1) \int \frac{\cos^{n-2} x\,dx}{x^{m-2}} \right].$$

**358.** $$\int x^p \sin^m x \cos^n x\,dx$$

$$= \frac{1}{(m+n)^2}\left[ x^{p-1} \sin^m x \cos^{n-1} x\,(p \cos x + (m+n)\, x \sin x) \right.$$

$$\left. + (n-1)(m+n) \int x^p \sin^m x \cos^{n-2} x\,dx \right.$$

$$- mp \int x^{p-1} \sin^{m-1} x \cos^{n-1} x \, dx$$

$$- p(p-1) \int x^{p-2} \sin^m x \cos^n x \, dx \Big].$$

$$= \frac{1}{(m+n)^2} \Big[ x^{p-1} \sin^{m-1} x \cos^n x (p \sin x - (m+n) x \cos x)$$

$$+ (m-1)(m+n) \int x^p \sin^{m-2} x \cos^n x \, dx$$

$$+ np \int x^{p-1} \sin^{m-1} x \cos^{n-1} x \, dx$$

$$- p(p-1) \int x^{p-2} \sin^m x \cos^n x \, dx \Big].$$

**359.** $\int \sin mx \sin nx \, dx = \frac{\sin (m-n)x}{2(m-n)} - \frac{\sin (m+n)x}{2(m+n)}.$

**360.** $\int \sin mx \cos nx \, dx = -\frac{\cos (m-n)x}{2(m-n)} - \frac{\cos (m+n)x}{2(m+n)}.$

**361.** $\int \cos mx \cos nx \, dx = \frac{\sin (m-n)x}{2(m-n)} + \frac{\sin (m+n)x}{2(m+n)}.$

See page 61.

**362.** $\int \sin^2 mx \, dx = \frac{1}{2m} (mx - \sin mx \cos mx).$

**363.** $\int \cos^2 mx \, dx = \frac{1}{2m} (mx + \sin mx \cos mx).$

**364.** $\int \sin mx \cos mx \, dx = -\frac{1}{4m} \cos 2mx.$

**365.** $\int \sin nx \sin^m x \, dx = \frac{1}{m+n} \Big[ - \cos nx \sin^m x$

$$+ m \int \cos (n-1)x \cdot \sin^{m-1} x \, dx \Big]$$

366. $$\int \sin nx \cos^m x\,dx = \frac{1}{m+n}\left[ -\cos nx \cos^m x + m\int \sin(n-1)x \cdot \cos^{m-1} x\,dx \right].$$

367. $$\int \cos nx \sin^m x\,dx = \frac{1}{m+n}\left[ \sin nx \sin^m x - m\int \sin(n-1)x \cdot \sin^{m-1} x\,dx \right].$$

368. $$\int \cos nx \cos^m x\,dx = \frac{1}{m+n}\left[ \sin nx \cos^m x + m\int \cos(n-1)x \cdot \cos^{m-1} x\,dx \right].$$

369. $$\int \frac{\cos nx\,dx}{\cos^m x} = 2\int \frac{\cos(n-1)x\,dx}{\cos^{m-1} x} - \int \frac{\cos(n-2)x\,dx}{\cos^m x}.$$

370. $$\int \frac{\cos nx\,dx}{\sin^m x} = -2\int \frac{\sin(n-1)x\,dx}{\sin^{m-1} x} + \int \frac{\cos(n-2)x\,dx}{\sin^m x}.$$

371. $$\int \frac{\sin nx\,dx}{\sin^m x} = 2\int \frac{\cos(n-1)x\,dx}{\sin^{m-1} x} + \int \frac{\sin(n-2)x\,dx}{\sin^m x}.$$

372. $$\int \frac{\sin nx\,dx}{\cos^m x} = 2\int \frac{\sin(n-1)x\,dx}{\cos^{m-1} x} - \int \frac{\sin(n-2)x\,dx}{\cos^m x}.$$

373. $$\int \frac{(\cos px + i \sin px)\,dx}{\cos nx} = -2i\int \frac{z^{p+n-1}\,dz}{1+z^{2n}},$$

where $z = \cos x + i \sin x$. This yields two real integrals.

374. $$\int \frac{(\cos px + i \sin px)\,dx}{\sin nx} = -2\int \frac{z^{p+n-1}\,dz}{1-z^{2n}},$$

where $z = \cos x + i \sin x$. This yields two real integrals.

**375.** $\int \frac{(i\cos x - \sin x)\,dx}{\sqrt[n]{\cos nx}} = \int \frac{dy}{2 - y^n},$

where $y = \frac{\cos x + i \sin x}{\sqrt[n]{\cos nx}}$. This yields two real integrals.

**376.** $\int \sin ax \sin bx \sin cx\,dx = -\tfrac{1}{4}\left\{\frac{\cos(a-b+c)x}{a-b+c}\right.$

$\left. + \frac{\cos(b+c-a)x}{b+c-a} + \frac{\cos(a+b-c)x}{a+b-c} - \frac{\cos(a+b+c)x}{a+b+c}\right\}.$

**377.** $\int \cos ax \cos bx \cos cx\,dx = \tfrac{1}{4}\left\{\frac{\sin(a+b+c)x}{a+b+c}\right.$

$\left. + \frac{\sin(b+c-a)x}{b+c-a} + \frac{\sin(a-b+c)x}{a-b+c} + \frac{\sin(a+b-c)x}{a+b-c}\right\}.$

**378.** $\int \sin ax \cos bx \cos cx\,dx = -\tfrac{1}{4}\left\{\frac{\cos(a+b+c)x}{a+b+c}\right.$

$\left. - \frac{\cos(b+c-a)x}{b+c-a} + \frac{\cos(a+b-c)x}{a+b-c} + \frac{\cos(a+c-b)x}{a+c-b}\right\}.$

**379.** $\int \cos ax \sin bx \sin cx\,dx = \tfrac{1}{4}\left\{\frac{\sin(a+b-c)x}{a+b-c}\right.$

$\left. + \frac{\sin(a-b+c)x}{a-b+c} - \frac{\sin(a+b+c)x}{a+b+c} - \frac{\sin(b+c-a)x}{b+c-a}\right\}.$

**380.** $\int \sin^{-1}x\,dx = x\sin^{-1}x + \sqrt{1-x^2}.$

**381.** $\int \cos^{-1}x\,dx = x\cos^{-1}x - \sqrt{1-x^2}.$

**382.** $\int \tan^{-1}x\,dx = x\tan^{-1}x - \tfrac{1}{2}\log(1+x^2).$

**383.** $\int \operatorname{ctn}^{-1}x\,dx = x\operatorname{ctn}^{-1}x + \tfrac{1}{2}\log(1+x^2).$

**384.** $\int \sec^{-1} x\,dx = x \sec^{-1} x - \log(x + \sqrt{x^2 - 1}).$

**385.** $\int \csc^{-1} x\,dx = x \csc^{-1} x + \log(x + \sqrt{x^2 - 1}).$

**386.** $\int \operatorname{versin}^{-1} x\,dx = (x - 1)\operatorname{versin}^{-1} x + \sqrt{2x - x^2}.$

**387.** $\int (\sin^{-1} x)^2\,dx = x(\sin^{-1} x)^2 - 2x + 2\sqrt{1 - x^2}\sin^{-1} x.$

**388.** $\int (\cos^{-1} x)^2\,dx = x(\cos^{-1} x)^2 - 2x - 2\sqrt{1 - x^2}\cos^{-1} x.$

**389.** $\int x \sin^{-1} x\,dx = \tfrac{1}{4}[(2x^2 - 1)\sin^{-1} x + x\sqrt{1 - x^2}].$

**390.** $\int x \cos^{-1} x\,dx = \tfrac{1}{4}[(2x^2 - 1)\cos^{-1} x - x\sqrt{1 - x^2}].$

**391.** $\int x \tan^{-1} x\,dx = \tfrac{1}{2}[(x^2 + 1)\tan^{-1} x - x].$

**392.** $\int x \operatorname{ctn}^{-1} x\,dx = \tfrac{1}{2}[(x^2 + 1)\operatorname{ctn}^{-1} x + x].$

**393.** $\int x \sec^{-1} x\,dx = \tfrac{1}{2}[x^2 \sec^{-1} x - \sqrt{x^2 - 1}].$

**394.** $\int x \csc^{-1} x\,dx = \tfrac{1}{2}[x^2 \csc^{-1} x + \sqrt{x^2 - 1}].$

**395.** $\int x^n \sin^{-1} x\,dx = \dfrac{1}{n+1}\left(x^{n+1}\sin^{-1} x - \int \dfrac{x^{n+1}\,dx}{\sqrt{1 - x^2}}\right).$

**396.** $\int x^n \cos^{-1} x\,dx = \dfrac{1}{n+1}\left(x^{n+1}\cos^{-1} x + \int \dfrac{x^{n+1}\,dx}{\sqrt{1 - x^2}}\right).$

397. $\int x^n \tan^{-1} x\,dx = \frac{1}{n+1}\left( x^{n+1}\tan^{-1}x - \int \frac{x^{n+1}\,dx}{1+x^2}\right).$

398. $\int x^n \operatorname{ctn}^{-1} x\,dx = \frac{1}{n+1}\left( x^{n+1}\operatorname{ctn}^{-1}x + \int \frac{x^{n+1}\,dx}{1+x^2}\right).$

399. $\int \frac{\sin^{-1}x\,dx}{x^2} = \log\left(\frac{1-\sqrt{1-x^2}}{x}\right) - \frac{\sin^{-1}x}{x}.$

400. $\int \frac{\tan^{-1}x\,dx}{x^2} = \log x - \tfrac{1}{2}\log(1+x^2) - \frac{\tan^{-1}x}{x}.$

401. $\int e^{ax}\,dx = \frac{e^{ax}}{a}. \qquad \int f(e^{ax})\,dx = \int \frac{f(y)\,dy}{ay},\ y = e^{ax}.$

402. $\int x e^{ax}\,dx = \frac{e^{ax}}{a^2}(ax-1).$

403. $\int x^m e^{ax}\,dx = \frac{x^m e^{ax}}{a} - \frac{m}{a}\int x^{m-1}e^{ax}\,dx.$

404. $\int \frac{e^{ax}}{x^m}\,dx = \frac{1}{m-1}\left[ -\frac{e^{ax}}{x^{m-1}} + a\int \frac{e^{ax}\,dx}{x^{m-1}}\right].$

405. $\int a^{bx}\,dx = \frac{a^{bx}}{b\log a}. \qquad \int f(a^{bx})\,dx = \int \frac{f(y)\,dy}{b\cdot\log a\cdot y},\ y = a^{bx}$

406. $\int x^n a^x\,dx = \frac{a^x x^n}{\log a} - \frac{n a^x x^{n-1}}{(\log a)^2} + \frac{n(n-1)\,a^x x^{n-2}}{(\log a)^3} \cdots$
$$\pm \frac{n(n-1)(n-2)\cdots 2.1\,a^x}{(\log a)^{n+1}}.$$

407. $\int \frac{a^x\,dx}{x^n} = \frac{1}{n-1}\left[ -\frac{a^x}{x^{n-1}} - \frac{a^x\cdot\log a}{(n-2)x^{n-2}}\right.$
$$\left. -\frac{a^x\cdot(\log a)^2}{(n-2)(n-3)x^{n-3}} - \cdots + \frac{(\log a)^{n-1}}{(n-2)(n-3)\cdots 2.1}\int \frac{a^x\,dx}{x}\right].$$

408. $\int \frac{a^x\,dx}{x} = \log x + x\log a + \frac{(x\log a)^2}{2\cdot 2!} + \frac{(x\log a)^3}{3\cdot 3!} + \cdots.$

409. $\int \frac{dx}{1+e^x} = \log \frac{e^x}{1+e^x}.$

410. $\int \frac{dx}{a+be^{mx}} = \frac{1}{am}[mx - \log(a + be^{mx})].$

411. $\int \frac{dx}{ae^{mx}+be^{-mx}} = \frac{1}{m\sqrt{ab}} \tan^{-1}\left(e^{mx}\sqrt{\frac{a}{b}}\right)$

412. $\int \frac{dx}{\sqrt{a+be^{mx}}} = \frac{-2}{m\sqrt{-a}} \sin^{-1}\sqrt{\frac{-a}{b}}\, e^{-\frac{1}{2}mx},$

or $\frac{-2}{m\sqrt{a}} \log(\sqrt{a} + \sqrt{a+be^{mx}}) + \frac{x}{\sqrt{a}}.$

413. $\int \frac{xe^x\,dx}{(1+x)^2} = \frac{e^x}{1+x}, \quad \int x^n \cdot e^{ax^{n+1}} dx = \frac{e^{ax^{n+1}}}{a(n+1)}$

414. $\int e^{ax} \sin px\,dx = \frac{e^{ax}(a \sin px - p \cos px)}{a^2+p^2}.$

415. $\int e^{ax} \cos px\,dx = \frac{e^{ax}(a \cos px + p \sin px)}{a^2+p^2}.$

416. $\int e^{ax} \log x\,dx = \frac{e^{ax} \log x}{a} - \frac{1}{a}\int \frac{e^{ax}dx}{x}.$

417. $\int e^{ax} \sin^2 x\,dx = \frac{e^{ax}}{4+a^2}\left(\sin x(a \sin x - 2\cos x) + \frac{2}{a}\right).$

418. $\int e^{ax} \cos^2 x\,dx = \frac{e^{ax}}{4+a^2}\left(\cos x(2 \sin x + a \cos x) + \frac{2}{a}\right).$

419. $\int e^{ax} \sin^n bx\,dx = \frac{1}{a^2+n^2b^2}\Big((a \sin bx$

$- nb \cos bx)\, e^{ax} \sin^{n-1} bx + n(n-1)b^2 \int e^{ax} \sin^{n-2} bx \cdot dx\Big).$

**420.** $$\int e^{ax}\cos^n bx\,dx = \frac{1}{a^2+n^2b^2}\Big((a\cos bx + nb\sin bx)\,e^{ax}\cos^{n-1}bx + n(n-1)b^2\int e^{ax}\cos^{n-2}bx\,dx\Big).$$

**421.** $$\int e^{ax}\tan^n x\,dx = \frac{e^{ax}\tan^{n-1}x}{n-1} - \frac{a}{n-1}\int e^{ax}\tan^{n-1}x\,dx - \int e^{ax}\tan^{n-2}x\,dx.$$

**422.** $$\int e^{ax}\operatorname{ctn}^n x\,dx = -\frac{e^{ax}\operatorname{ctn}^{n-1}x}{n-1} + \frac{a}{n-1}\int e^{ax}\operatorname{ctn}^{n-1}x\,dx - \int e^{ax}\operatorname{ctn}^{n-2}x\,dx.$$

**423.** $$\int\frac{e^{ax}\,dx}{\sin^n x} = -e^{ax}\frac{a\sin x+(n-2)\cos x}{(n-1)(n-2)\sin^{n-1}x} + \frac{a^2+(n-2)^2}{(n-1)(n-2)}\int\frac{e^{ax}\,dx}{\sin^{n-2}x}.$$

**424.** $$\int\frac{e^{ax}\,dx}{\cos^n x} = -e^{ax}\frac{a\cos x-(n-2)\sin x}{(n-1)(n-2)\cos^{n-1}x} + \frac{a^2+(n-2)^2}{(n-1)(n-2)}\int\frac{e^{ax}\,dx}{\cos^{n-2}x}.$$

**425.** $$\int e^{ax}\sin^m x\cos^n x\,dx = \frac{1}{(m+n)^2+a^2}\Big\{e^{ax}\sin^m x\cos^{n-1}x\,(a\cos x+(m+n)\sin x) - ma\int e^{ax}\sin^{m-1}x\cos^{n-1}x\,dx + (n-1)(m+n)\int e^{ax}\sin^m x\cos^{n-2}x\,dx\Big\}$$

$$= \frac{1}{(m+n)^2+a^2}\Big\{ e^{ax}\sin^{m-1}x\cos^n x(a\sin x-(m+n)\cos x)$$
$$+ na\int e^{ax}\sin^{m-1}x\cos^{n-1}x\,dx$$
$$+ (m-1)(m+n)\int e^{ax}\sin^{m-2}x\cos^n x\,dx\Big\}$$
$$= \frac{1}{(m+n)^2+a^2}\Big\{ [e^{ax}\cos^{n-1}x\sin^{m-1}x(a\sin x\cos x+n\sin^2 x$$
$$- m\cos^2 x)] + n(n-1)\int e^{ax}\sin^m x\cos^{n-2}x\,dx$$
$$+ m(m-1)\int e^{ax}\sin^{m-2}x\cos^n x\,dx\Big\}$$
$$= \frac{1}{(m+n)^2+a^2}\Big\{ [e^{ax}\sin^{m-1}x\cos^{n-1}x(a\sin x\cos x+n\sin^2 x$$
$$- m\cos^2 x)] + n(n-1)\int e^{ax}\sin^{m-2}x\cos^{n-2}x\,dx$$
$$+ (m-n)(m+n-1)\int e^{ax}\sin^{m-2}x\cos^n x\,dx\Big\}$$
$$= \frac{1}{(m+n)^2+a^2}\Big\{ [e^{ax}\sin^{m-1}x\cos^{n-1}x(a\sin x\cos x+n\sin^2 x$$
$$- m\cos^2 x)] + m(m-1)\int e^{ax}\sin^{m-2}x\cos^{n-2}x\,dx$$
$$- (m-n)(m+n-1)\int e^{ax}\sin^m x\cos^{n-2}x\,dx\Big\}.$$

**426.** $\int \log x\,dx = x\log x - x.$

**427.** $\int x^m \log x\,dx = x^{m+1}\left[\frac{\log x}{m+1} - \frac{1}{(m+1)^2}\right].$

**428.** $\int (\log x)^n dx = x(\log x)^n - n\int (\log x)^{n-1}dx.$

**429.** $\int x^m (\log x)^n dx = \frac{x^{m+1}(\log x)^n}{m+1} - \frac{n}{m+1}\int x^m (\log x)^{n-1}dx.$

430. $\int \frac{(\log x)^n dx}{x} = \frac{(\log x)^{n+1}}{n+1}.$

431. $\int \frac{dx}{\log x} = \log(\log x) + \log x + \frac{(\log x)^2}{2 \cdot 2!} + \frac{(\log x)^3}{3 \cdot 3!} + \cdots.$

432. $\int \frac{dx}{(\log x)^n} = -\frac{x}{(n-1)(\log x)^{n-1}} + \frac{1}{n-1}\int \frac{dx}{(\log x)^{n-1}}.$

433. $\int \frac{x^m dx}{(\log x)^n} = -\frac{x^{m+1}}{(n-1)(\log x)^{n-1}} + \frac{m+1}{n-1}\int \frac{x^m dx}{(\log x)^{n-1}}.$

434. $\int \frac{x^m dx}{\log x} = \int \frac{e^{-y}}{y} dy,$ where $y = -(m+1)\log x.$

435. $\int \frac{dx}{x \log x} = \log(\log x),$ and $\int \frac{(n-1)dx}{x(\log x)^n} = \frac{-1}{(\log x)^{n-1}}.$

436. $\int \log(a^2 + x^2)\, dx = x \cdot \log(a^2 + x^2) - 2x + 2a \tan^{-1}\left(\frac{x}{a}\right)$

437. $\int (a + bx)^m \log x\, dx$

$$= \frac{1}{b(m+1)}\left[(a+bx)^{m+1}\log x - \int \frac{(a+bx)^{m+1}dx}{x}\right].$$

438. $\int x^m \log(a + bx)\, dx$

$$= \frac{1}{m+1}\left[x^{m+1}\log(a+bx) - b\int \frac{x^{m+1}dx}{a+bx}\right].$$

439. $\int \frac{\log(a+bx)\,dx}{x}$

$$= \log a \cdot \log x + \frac{bx}{a} - \frac{1}{2^2}\left(\frac{bx}{a}\right)^2 + \frac{1}{3^2}\left(\frac{bx}{a}\right)^3 - \cdots$$

$$= \tfrac{1}{2}(\log bx)^2 - \frac{a}{bx} + \frac{1}{2^2}\left(\frac{a}{bx}\right)^2 - \frac{1}{3^2}\left(\frac{a}{bx}\right)^3 + \cdots.$$

**440.** $\int \frac{\log x\,dx}{(a+bx)^m}$

$$= \frac{1}{b(m-1)}\left[-\frac{\log x}{(a+bx)^{m-1}} + \int \frac{dx}{x(a+bx)^{m-1}}\right].$$

**441.** $\int \frac{\log x\,dx}{a+bx} = \frac{1}{b}\log x \cdot \log(a+bx) - \frac{1}{b}\int \frac{\log(a+bx)\,dx}{x}.$

**442.** $\int (a+bx)\log x\,dx = \frac{(a+bx)^2}{2b}\log x - \frac{a^2\log x}{2b} - ax - \frac{1}{4}bx^2.$

**443.** $\int \frac{\log x\,dx}{\sqrt{a+bx}}$

$$= \frac{2}{b}\left[(\log x - 2)\sqrt{a+bx} + \sqrt{a}\,\log(\sqrt{a+bx}+\sqrt{a})\right.$$
$$\left. -\sqrt{a}\,\log(\sqrt{a+bx}-\sqrt{a})\right], \text{ if } a>0$$

$$= \frac{2}{b}\left[(\log x - 2)\sqrt{a+bx} + 2\sqrt{-a}\,\tan^{-1}\sqrt{\frac{a+bx}{-a}}\right], \text{ if } a<0.$$

**444.** $\int \sin\log x\,dx = \frac{1}{2}x[\sin\log x - \cos\log x].$

**445.** $\int \cos\log x\,dx = \frac{1}{2}x[\sin\log x + \cos\log x].$

**446.** $\int \sinh x\,dx = \cosh x.$

**447.** $\int \cosh x\,dx = \sinh x.$

**448.** $\int \tanh x\,dx = \log\cosh x.$

**449.** $\int \operatorname{ctnh} x\,dx = \log\sinh x.$

**450.** $\int \operatorname{sech} x\,dx = 2\tan^{-1} e^x.$

**451.** $\int \operatorname{csch} x\,dx = \log \tanh \frac{x}{2}.$

**452.** $\int \sinh^n x\,dx = \frac{1}{n}\sinh^{n-1} x \cdot \cosh x - \frac{n-1}{n}\int \sinh^{n-2} x\,dx$

$$= \frac{1}{n+1}\sinh^{n+1} x \cosh x - \frac{n+2}{n+1}\int \sinh^{n+2} x\,dx.$$

**453.** $\int \cosh^n x\,dx = \frac{1}{n}\sinh x \cdot \cosh^{n-1} x + \frac{n-1}{n}\int \cosh^{n-2} x\,dx$

$$= -\frac{1}{n+1}\sinh x \cosh^{n+1} x + \frac{n+2}{n+1}\int \cosh^{n+2} x\,dx$$

**454.** $\int x \sinh x\,dx = x \cosh x - \sinh x.$

**455.** $\int x \cosh x\,dx = x \sinh x - \cosh x.$

**456.** $\int x^2 \sinh x\,dx = (x^2+2)\cosh x - 2x \sinh x.$

**457.** $\int x^n \sinh x\,dx = x^n \cosh x - nx^{n-1} \sinh x$

$$+ n(n-1)\int x^{n-2} \sinh x\,dx.$$

**458.** $\int \sinh^2 x\,dx = \tfrac{1}{2}(\sinh x \cosh x - x).$

**459.** $\int \sinh x \cdot \cosh x\,dx = \tfrac{1}{4}\cosh(2x).$

**460.** $\int \cosh^2 x\,dx = \tfrac{1}{2}(\sinh x \cosh x + x).$

**461.** $\int \tanh^2 x\,dx = x - \tanh x.$

462. $\int \operatorname{ctnh}^2 x\,dx = x - \operatorname{ctnh} x.$

463. $\int \operatorname{sech}^2 x\,dx = \tanh x.$

464. $\int \operatorname{csch}^2 x\,dx = -\operatorname{ctnh} x.$

465. $\int \sinh^{-1} x\,dx = x \sinh^{-1} x - \sqrt{1+x^2}.$

466. $\int \cosh^{-1} x\,dx = x \cosh^{-1} x - \sqrt{x^2-1}.$

467. $\int \tanh^{-1} x\,dx = x \tanh^{-1} x + \tfrac{1}{2}\log(1-x^2).$

468. $\int x \sinh^{-1} x\,dx = \tfrac{1}{4}[(2x^2+1)\sinh^{-1} x - x\sqrt{1+x^2}].$

469. $\int x \cosh^{-1} x\,dx = \tfrac{1}{4}[(2x^2-1)\cosh^{-1} x - x\sqrt{x^2-1}].$

470. $\int \frac{dx}{\cosh a + \cosh x}$

$= \operatorname{csch} a\,[\log\cosh \tfrac{1}{2}(x+a) - \log\cosh \tfrac{1}{2}(x-a)],$

$= 2\operatorname{csch} a \cdot \tanh^{-1}(\tanh \tfrac{1}{2}x \cdot \tanh \tfrac{1}{2}a).$

471. $\int \frac{dx}{\cos a + \cosh x} = 2\csc a \cdot \tan^{-1}(\tanh \tfrac{1}{2}x \cdot \tan \tfrac{1}{2}a).$

472. $\int \frac{dx}{1 + \cos a \cdot \cosh x} = 2\csc a \cdot \tanh^{-1}(\tanh \tfrac{1}{2}x \cdot \tan \tfrac{1}{2}a).$

473. $\int \sinh x \cdot \cos x\,dx = \tfrac{1}{2}(\cosh x \cdot \cos x + \sinh x \cdot \sin x).$

474 $\int \cosh x \cdot \cos x\,dx = \tfrac{1}{2}(\sinh x \cdot \cos x + \cosh x \cdot \sin x).$

475. $\int \sinh x \cdot \sin x\,dx = \tfrac{1}{2}(\cosh x \cdot \sin x - \sinh x \cdot \cos x).$

**476.** $\int \cosh x \cdot \sin x\,dx = \frac{1}{2}(\sinh x \cdot \sin x - \cosh x \cdot \cos x).$

**477.** $\int \sinh(mx)\sinh(nx)\,dx$

$$= \frac{1}{m^2 - n^2}\Big[m \sinh(nx)\cosh(mx) - n\cosh(nx)\sinh(mx)\Big].$$

**478.** $\int \cosh(mx)\sinh(nx)\,dx$

$$= \frac{1}{m^2 - n^2}\Big[m \sinh(nx)\sinh(mx) - n\cosh(nx)\cosh(mx)\Big].$$

**479.** $\int \cosh(mx)\cosh(nx)\,dx$

$$= \frac{1}{m^2 - n^2}\Big[m \sinh(mx)\cosh(nx) - n\sinh(nx)\cosh(mx)\Big].$$

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$$\int \frac{dx}{a\cos^2 x + c\sin x\cdot\cos x + b\sin^2 x} = \int \frac{d(\tan x)}{a + c\tan x + b\tan^2 x}.$$

$$\int \frac{(l + m\cos x + n\sin x)\,dx}{a + b\cos x + c\sin x} = \int \frac{(m\cos\delta + n\sin\delta)\cos z\cdot dz}{Z} + \int\int \frac{l\cdot dz}{Z} - \int \frac{(m\sin\delta - n\cos\delta)\sin z\cdot dz}{Z},$$

where $b = q\cdot\cos\delta$, $c = q\cdot\sin\delta$, $z = x - \delta$, $Z = a + q\cdot\cos z$.

$\int \sin(mx + a)\cdot\sin(nx + b)\,dx$ [See 303 and 304.]

$$= \frac{\sin[mx - nx + a - b]}{2(m - n)} - \frac{\sin[mx + nx + a + b]}{2(m + n)}.$$

$\int \cos(mx + a)\cdot\cos(nx + b)\,dx$

$$= \frac{\sin[mx + nx + a + b]}{2(m + n)} + \frac{\sin[mx - nx + a - b]}{2(m - n)}.$$

$\int \sin(mx + a)\cdot\cos(nx + b)\,dx$

$$= -\frac{\cos[mx + nx + a + b]}{2(m + n)} - \frac{\cos[mx - nx + a - b]}{2(m - n)}.$$

## VI. MISCELLANEOUS DEFINITE INTEGRALS.*

480. $\int_0^\infty \frac{a\,dx}{a^2+x^2} = \frac{\pi}{2}$, if $a>0$; 0, if $a=0$; $-\frac{\pi}{2}$, if $a<0$.

481. $\int_0^\infty x^{n-1}e^{-x}\,dx = \int_0^1 \left[\log\frac{1}{x}\right]^{n-1} dx \equiv \Gamma(n)$.

$$\Gamma(z+1) = z\cdot\Gamma(z), \text{ if } z>0.$$

$\Gamma(y)\cdot\Gamma(1-y) = \frac{\pi}{\sin \pi y}$, if $1>y>0$. $\Gamma(2)=\Gamma(1)=1$.

$\Gamma(n+1)=n!$, if $n$ is an integer. $\Gamma(z)=\Pi(z-1)$.

$\Gamma(\frac{1}{2})=\sqrt{\pi}$. $Z(y)=D_y[\log\Gamma(y)]$. $Z(1)=-0.577216$.

482. $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \int_0^\infty \frac{x^{m-1}\,dx}{(1+x)^{m+n}} = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$.

483. $\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$

$= \frac{1\cdot3\cdot5\cdots(n-1)}{2\cdot4\cdot6\cdots(n)}\cdot\frac{\pi}{2}$, if $n$ is an even integer,

$= \frac{2\cdot4\cdot6\cdots(n-1)}{1\cdot3\cdot5\cdot7\cdots n}$, if $n$ is an odd integer,

$= \frac{1}{2}\sqrt{\pi}\,\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}$, for any value of $n$ greater than $-1$.

484. $\int_0^\infty \frac{\sin mx\,dx}{x} = \frac{\pi}{2}$, if $m>0$; 0, if $m=0$; $-\frac{\pi}{2}$, if $m<0$.

* For very complete lists of definite integrals, see Bierens de Haan, *Tables d'intégrales définies*, Amsterdam, 1858-64, and *Nouv. Tables d'intégrales définies*, Leyden, 1867.

485. $\int_0^\infty \frac{\sin x \cdot \cos mx\, dx}{x} = 0$, if $m < -1$ or $m > 1$;

$\frac{\pi}{4}$, if $m = -1$ or $m = 1$; $\frac{\pi}{2}$, if $-1 < m < 1$.

486. $\int_0^\infty \frac{\sin^2 x\, dx}{x^2} = \frac{\pi}{2}$.

487. $\int_0^\infty \cos(x^2)\, dx = \int_0^\infty \sin(x^2)\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$.

488. $\int_0^\pi \sin kx \cdot \sin mx\, dx = \int_0^\pi \cos kx \cdot \cos mx\, dx = 0$,

if $k$ is different from $m$.

489. $\int_0^\pi \sin^2 mx\, dx = \int_0^\pi \cos^2 mx\, dx = \frac{\pi}{2}$.

490. $\int_0^\infty \frac{\cos mx\, dx}{1+x^2} = \frac{\pi}{2} \cdot e^{-|m|}$. $m > 0$.

491. $\int_0^\infty \frac{\cos x\, dx}{\sqrt{x}} = \int_0^\infty \frac{\sin x\, dx}{\sqrt{x}} = \sqrt{\frac{\pi}{2}}$.

492. $\int_0^\infty e^{-a^2x^2}\, dx = \frac{1}{2a}\sqrt{\pi} \cdot = \frac{1}{2a}\Gamma(\frac{1}{2})$. $a > 0$

493. $\int_0^\infty x^n e^{-ax}\, dx = \frac{\Gamma(n+1)}{a^{n+1}} = \frac{n!}{a^{n+1}}$. $n > -1$, $a > 0$.

494. $\int_0^\infty x^{2n} e^{-ax^2}\, dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^{n+1} a^n}\sqrt{\frac{\pi}{a}}$.

495. $\int_0^\infty e^{-x^2 - \frac{a^2}{x^2}}\, dx = \frac{e^{-2a}\sqrt{\pi}}{2}$. $a > 0$

496. $\int_0^\infty e^{-nx}\sqrt{x}\, dx = \frac{1}{2n}\sqrt{\frac{\pi}{n}}$.

497. $\int_0^\infty \frac{e^{-nx}}{\sqrt{x}}\, dx = \sqrt{\frac{\pi}{n}}$.

498. $\int_0^\infty \frac{dx}{e^{nx}+e^{-nx}} = \frac{\pi}{4n}.$

499. $\int_0^\infty \frac{x\,dx}{e^{nx}-e^{-nx}} = \frac{\pi^2}{8n^2}.$

500. $\int_0^{\pi i} \sinh(mx)\cdot\sinh(nx)\,dx = \int_0^{\pi i} \cosh(mx)\cdot\cosh(nx)\,dx$
$= 0$, if $m$ is different from $n$.

501. $\int_0^{\pi i} \cosh^2(mx)\,dx = -\int_0^{\pi i} \sinh^2(mx)\,dx = \frac{\pi i}{2}.$

502. $\int_{-\pi i}^{+\pi i} \sinh(mx)\,dx = 0.$

503. $\int_0^{\pi i} \cosh(mx)\,dx = 0.$

504. $\int_{-\pi i}^{\pi i} \sinh(mx)\cosh(nx)\,dx = 0.$

505. $\int_0^{\pi i} \sinh(mx)\cosh(mx)\,dx = 0.$

506. $\int_0^\infty e^{-ax}\cos mx\,dx = \frac{a}{a^2+m^2}$, if $a>0$.

507. $\int_0^\infty e^{-ax}\sin mx\,dx = \frac{m}{a^2+m^2}$, if $a>0$.

508. $\int_0^\infty e^{-a^2x^2}\cos bx\,dx = \frac{\sqrt{\pi}\cdot e^{-\frac{b^2}{4a^2}}}{2a}.$ $a>0$.

509. $\int_0^1 \frac{\log x}{1-x}\,dx = -\frac{\pi^2}{6}.$

510. $\int_0^1 \frac{\log x}{1+x}\,dx = -\frac{\pi^2}{12}.$

511. $\int_0^1 \frac{\log x}{1-x^2}\,dx = -\frac{\pi^2}{8}.$

512. $\int_0^1 \log\left(\frac{1+x}{1-x}\right)\cdot\frac{dx}{x} = \frac{\pi^2}{4}.$

513. $\int_0^1 \frac{\log x\,dx}{\sqrt{1-x^2}} = -\frac{\pi}{2}\log 2.$

514. $\int_0^1 \frac{(x^p - x^q)\,dx}{\log x} = \log\frac{p+1}{q+1}$, if $p+1>0$, $q+1>0$.

515. $\int_0^1 (\log x)^n dx = (-1)^n \cdot n!.$

516. $\int_0^1 \left(\log\frac{1}{x}\right)^{\frac{1}{2}} dx = \frac{\sqrt{\pi}}{2}.$

517. $\int_0^1 \left(\log\frac{1}{x}\right)^n dx = n!.$

518. $\int_0^1 \frac{dx}{\sqrt{\log\left(\frac{1}{x}\right)}} = \sqrt{\pi}.$

519. $\int_0^1 x^m \left(\log\frac{1}{x}\right)^n dx = \frac{\Gamma(n+1)}{(m+1)^{n+1}}$, if $m+1>0$, $n+1>0$

520. $\int_0^\infty \log\left(\frac{e^x+1}{e^x-1}\right)dx = \frac{\pi^2}{4}.$

521. $\int_0^{\frac{\pi}{2}} \log\sin x\,dx = \int_0^{\frac{\pi}{2}} \log\cos x\,dx = -\frac{\pi}{2}\cdot\log 2.$

522. $\int_0^\pi x\cdot\log\sin x\,dx = -\frac{\pi^2}{2}\log 2.$

523. $\int_0^\pi \log(a \pm b\cos x)\,dx = \pi\log\left(\frac{a+\sqrt{a^2-b^2}}{2}\right).$ $\quad a \geqq b.$

## VII. ELLIPTIC INTEGRALS.

$$F(\phi, k) \equiv \int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}} = \int_0^{x} \frac{dz}{\sqrt{1 - z^2}\sqrt{1 - k^2 z^2}} \equiv u,$$

where $k^2 < 1$, $x = \sin\phi$.

$$E(\phi, k) \equiv \int_0^{\phi} \sqrt{1 - k^2 \sin^2\theta}\cdot d\theta.$$

$$\Pi(\phi, n, k) \equiv \int_0^{\phi} \frac{d\theta}{(1 + n\sin^2\theta)\sqrt{1 - k^2\sin^2\theta}}.$$

$\phi \equiv \operatorname{am} u$, $\sin\phi = x \equiv \operatorname{sn} u$, $\cos\phi \equiv \sqrt{1 - x^2} \equiv \operatorname{cn} u$, $\tan\phi \equiv \operatorname{tn} u$,

$\Delta\phi \equiv \sqrt{1 - k^2\sin^2\phi} \equiv \sqrt{1 - k^2x^2} \equiv \operatorname{dn} u$, $k'^2 \equiv 1 - k^2$.

$$u \equiv \operatorname{am}^{-1}(\phi, k) \equiv \operatorname{sn}^{-1}(x, k) \equiv \operatorname{cn}^{-1}(\sqrt{1 - x^2}, k)$$
$$\equiv \operatorname{dn}^{-1}(\sqrt{1 - k^2x^2}, k).$$

$K \equiv F(\frac{1}{2}\pi, k)$, $K' \equiv F(\frac{1}{2}\pi, k')$, $E \equiv E(\frac{1}{2}\pi, k)$, $E' \equiv E(\frac{1}{2}\pi, k')$.

If $k_0 = \dfrac{2\sqrt{k}}{1 + k}$ and $\tan\phi \equiv \dfrac{\sin 2\omega}{k + \cos 2\omega}$,

$$F(\phi, k) \equiv \frac{2}{1 + k} F(\omega, k_0).$$

**524.** $$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}}$$

$$= \frac{\pi}{2}\left[1 + (\tfrac{1}{2})^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 k^6 + \cdots\right], \text{ if } k^2 < 1,$$

$$= K.$$

**525.** $$\int_0^{\frac{\pi}{2}} \sqrt{1 - k^2\sin^2\theta}\cdot d\theta$$

$$= \frac{\pi}{2}\left[1 - (\tfrac{1}{2})^2 k^2 - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \frac{k^4}{3} - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 \frac{k^6}{5} - \cdots\right], \text{ if } k^2 < 1,$$

$$= E.$$

526. $$\int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \frac{2}{\pi}\phi \cdot K - \sin\phi\cos\phi\left[\frac{1\cdot1}{2\cdot2}k^2 + \frac{1\cdot3}{2\cdot4}A_4k^4 + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}A_6k^6 + \cdots\right] = F(\phi, k),$$

where $A_4 \equiv \frac{1}{4}\sin^2\phi + \frac{3}{2\cdot4^2}$, $A_6 \equiv \frac{1}{6}\sin^4\phi + \frac{5}{6\cdot4}\sin^2\phi + \frac{5\cdot3}{6\cdot4\cdot2}$, $A_8 \equiv \frac{1}{8}\sin^6\phi + \frac{7}{8\cdot6}\sin^4\phi + \frac{7\cdot5}{8\cdot6\cdot4}\sin^2\phi + \frac{7\cdot5\cdot3}{8\cdot6\cdot4\cdot2}$, etc.

527. $$\int_0^{\phi} \sqrt{1-k^2\sin^2\theta}\cdot d\theta = \frac{2}{\pi}\phi \cdot E + \sin\phi\cos\phi\left[\frac{1\cdot1}{2\cdot2}k^2 + \frac{1}{2\cdot4}k^4A_4 + \frac{1\cdot3}{2\cdot4\cdot6}k^6A_6 + \cdots\right] = E(\phi, k).$$

528.* $$\int_0^{x} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \mathrm{sn}^{-1}(x, k) = F(\sin^{-1}x, k). \qquad 0 < x < 1.$$

529. $$\int_x^{1} \frac{dx}{\sqrt{(1-x^2)(k'^2+k^2x^2)}} = \mathrm{cn}^{-1}(x, k) = F(\cos^{-1}x, k) = \mathrm{sn}^{-1}(\sqrt{1-x^2}, k). \qquad 0 < x < 1.$$

530. $$\int_x^{1} \frac{dx}{\sqrt{(1-x^2)(x^2-k'^2)}} = \mathrm{dn}^{-1}(x, k) = F(\Delta^{-1}x, k) = \mathrm{sn}^{-1}\left(\frac{1}{k}\sqrt{1-x^2}, k\right). \qquad 0 < x < 1.$$

531. $$\int_0^{x} \frac{dx}{\sqrt{(1+x^2)(1+k'^2x^2)}} = \mathrm{tn}^{-1}(x, k) = F(\tan^{-1}x, k) = \mathrm{sn}^{-1}\left(\frac{x}{\sqrt{1+x^2}}, k\right). \qquad 0 < x < 1.$$

* The next forty-two integrals are copied in order from a class-room list of Prof. W. E. Byerly.

532. $$\int_0^x \frac{dx}{\sqrt{x(1-x)(1-k^2x)}} = 2\,\mathrm{sn}^{-1}(\sqrt{x},\,k)$$
$$= 2\,F(\sin^{-1}\sqrt{x},\,k). \quad 0<x<1.$$

533. $$\int_x^1 \frac{dx}{\sqrt{x(1-x)(k'^2+k^2x)}} = 2\,\mathrm{cn}^{-1}(\sqrt{x},\,k)$$
$$= 2\,F(\cos^{-1}\sqrt{x},\,k) = 2\,\mathrm{sn}^{-1}(\sqrt{1-x},\,k). \quad 0<x<1.$$

534. $$\int_x^1 \frac{dx}{\sqrt{x(1-x)(x-k'^2)}} = 2\,\mathrm{dn}^{-1}(\sqrt{x},\,k)$$
$$= 2\,F(\Delta^{-1}\sqrt{x},\,k) = 2\,\mathrm{sn}^{-1}\left(\frac{1}{k}\sqrt{1-x},\,k\right). \quad 0<x<1.$$

535. $$\int_0^x \frac{dx}{\sqrt{x(1+x)(1+k'^2x)}} = 2\,\mathrm{tn}^{-1}(\sqrt{x},\,k)$$
$$= 2\,F(\tan^{-1}\sqrt{x},\,k) = 2\,\mathrm{sn}^{-1}\left(\sqrt{\frac{x}{1+x}},\,k\right). \quad 0<x<1.$$

536. $$\int_0^x \frac{dx}{\sqrt{(a^2-x^2)(b^2-x^2)}} = \frac{1}{a}\,\mathrm{sn}^{-1}\left(\frac{x}{b},\,\frac{b}{a}\right). \quad a>b>x>0.$$

537. $$\int_x^\infty \frac{dx}{\sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{1}{a}\,\mathrm{sn}^{-1}\left(\frac{a}{x},\,\frac{b}{a}\right). \quad x>a>b.$$

538. $$\int_x^b \frac{dx}{\sqrt{(a^2+x^2)(b^2-x^2)}}$$
$$= \frac{1}{\sqrt{a^2+b^2}}\,\mathrm{cn}^{-1}\left(\frac{x}{b},\,\frac{b}{\sqrt{a^2+b^2}}\right). \quad b>x>0.$$

539. $$\int_b^x \frac{dx}{\sqrt{(a^2+x^2)(x^2-b^2)}}$$
$$= \frac{1}{\sqrt{a^2+b^2}}\,\mathrm{cn}^{-1}\left(\frac{b}{x},\,\frac{a}{\sqrt{a^2+b^2}}\right). \quad x>b>0.$$

540. $$\int_x^a \frac{dx}{\sqrt{(a^2 - x^2)(x^2 - b^2)}} = \frac{1}{a}\,\mathrm{sn}^{-1}\left(\sqrt{\frac{a^2 - x^2}{a^2 - b^2}},\ \sqrt{\frac{a^2 - b^2}{a^2}}\right). \qquad a > x > b.$$

541. $$\int_0^x \frac{dx}{\sqrt{(x^2 + a^2)(x^2 + b^2)}} = \frac{1}{a}\,\mathrm{tn}^{-1}\left(\frac{x}{b},\ \sqrt{\frac{a^2 - b^2}{a^2}}\right). \qquad x > 0.$$

$$a > \beta > \gamma.$$

542. $$\int_x^\infty \frac{dx}{\sqrt{(x - a)(x - \beta)(x - \gamma)}} = \frac{2}{\sqrt{a - \gamma}}\,\mathrm{sn}^{-1}\left(\sqrt{\frac{a - \gamma}{x - \gamma}},\ \sqrt{\frac{\beta - \gamma}{a - \gamma}}\right). \qquad x > a.$$

543. $$\int_a^x \frac{dx}{\sqrt{(x - a)(x - \beta)(x - \gamma)}} = \frac{2}{\sqrt{a - \gamma}}\,\mathrm{sn}^{-1}\left(\sqrt{\frac{x - a}{x - \beta}},\ \sqrt{\frac{\beta - \gamma}{a - \gamma}}\right). \qquad x > a.$$

544. $$\int_x^a \frac{dx}{\sqrt{(a - x)(x - \beta)(x - \gamma)}} = \frac{2}{\sqrt{a - \gamma}}\,\mathrm{sn}^{-1}\left(\sqrt{\frac{a - x}{a - \beta}},\ \sqrt{\frac{a - \beta}{a - \gamma}}\right). \qquad a > x > \beta.$$

545. $$\int_\beta^x \frac{dx}{\sqrt{(a - x)(x - \beta)(x - \gamma)}} = \frac{2}{\sqrt{a - \gamma}}\,\mathrm{sn}^{-1}\left(\sqrt{\frac{a - \gamma}{a - \beta}\cdot\frac{x - \beta}{x - \gamma}},\ \sqrt{\frac{a - \beta}{a - \gamma}}\right). \qquad a > x > \beta.$$

546. $$\int_x^\beta \frac{dx}{\sqrt{(a - x)(\beta - x)(x - \gamma)}} = \frac{2}{\sqrt{a - \gamma}}\,\mathrm{sn}^{-1}\left(\sqrt{\frac{a - \gamma}{\beta - \gamma}\cdot\frac{\beta - x}{a - x}},\ \sqrt{\frac{\beta - \gamma}{a - \gamma}}\right). \qquad \beta > x > \gamma.$$

547. $$\int_{\gamma}^{x} \frac{dx}{\sqrt{(a-x)(\beta-x)(x-\gamma)}}$$
$$= \frac{2}{\sqrt{a-\gamma}} \operatorname{sn}^{-1}\left(\sqrt{\frac{x-\gamma}{\beta-\gamma}}, \sqrt{\frac{\beta-\gamma}{a-\gamma}}\right). \qquad \beta > x > \gamma.$$

548. $$\int_{x}^{\gamma} \frac{dx}{\sqrt{(a-x)(\beta-x)(\gamma-x)}}$$
$$= \frac{2}{\sqrt{a-\gamma}} \operatorname{sn}^{-1}\left(\sqrt{\frac{\gamma-x}{\beta-x}}, \sqrt{\frac{a-\beta}{a-\gamma}}\right). \qquad \gamma > x.$$

549. $$\int_{-\infty}^{x} \frac{dx}{\sqrt{(a-x)(\beta-x)(\gamma-x)}}$$
$$= \frac{2}{\sqrt{a-\gamma}} \operatorname{sn}^{-1}\left(\sqrt{\frac{a-\gamma}{a-x}}, \sqrt{\frac{a-\beta}{a-\gamma}}\right). \qquad \gamma > x.$$

$$a > \beta > \gamma > \delta.$$

550. $$\int_{a}^{x} \frac{dx}{\sqrt{(x-a)(x-\beta)(x-\gamma)(x-\delta)}}$$
$$= \frac{2}{\sqrt{(a-\gamma)(\beta-\delta)}} \operatorname{sn}^{-1}\left(\sqrt{\frac{\beta-\delta}{a-\delta}\cdot\frac{x-a}{x-\beta}}, \sqrt{\frac{\beta-\gamma}{a-\gamma}\cdot\frac{a-\delta}{\beta-\delta}}\right).$$
$$x > a.$$

551. $$\int_{x}^{a} \frac{dx}{\sqrt{(a-x)(x-\beta)(x-\gamma)(x-\delta)}}$$
$$= \frac{2}{\sqrt{(a-\gamma)(\beta-\delta)}} \operatorname{sn}^{-1}\left(\sqrt{\frac{\beta-\delta}{a-\beta}\cdot\frac{a-x}{x-\delta}}, \sqrt{\frac{a-\beta}{a-\gamma}\cdot\frac{\gamma-\delta}{\beta-\delta}}\right).$$
$$a > x > \beta.$$

552. $$\int_{\beta}^{x} \frac{dx}{\sqrt{(a-x)(x-\beta)(x-\gamma)(x-\delta)}}$$
$$= \frac{2}{\sqrt{(a-\gamma)(\beta-\delta)}} \operatorname{sn}^{-1}\left(\sqrt{\frac{a-\gamma}{a-\beta}\cdot\frac{x-\beta}{x-\gamma}}, \sqrt{\frac{a-\beta}{a-\gamma}\cdot\frac{\gamma-\delta}{\beta-\delta}}\right).$$
$$a > x > \beta.$$

553. $$\int_x^\beta \frac{dx}{\sqrt{(a-x)(\beta-x)(x-\gamma)(x-\delta)}}$$

$$= \frac{2}{\sqrt{(a-\gamma)(\beta-\delta)}} \operatorname{sn}^{-1}\left(\sqrt{\frac{a-\gamma}{\beta-\gamma}\cdot\frac{\beta-x}{a-x}},\ \sqrt{\frac{\beta-\gamma}{a-\gamma}\cdot\frac{a-\delta}{\beta-\delta}}\right).$$

$\beta > x > \gamma.$

554. $$\int_\gamma^x \frac{dx}{\sqrt{(a-x)(\beta-x)(x-\gamma)(x-\delta)}}$$

$$= \frac{2}{\sqrt{(a-\gamma)(\beta-\delta)}} \operatorname{sn}^{-1}\left(\sqrt{\frac{\beta-\delta}{\beta-\gamma}\cdot\frac{x-\gamma}{x-\delta}},\ \sqrt{\frac{\beta-\gamma}{a-\gamma}\cdot\frac{a-\delta}{\beta-\delta}}\right).$$

$\beta > x > \gamma.$

555. $$\int_x^\gamma \frac{dx}{\sqrt{(a-x)(\beta-x)(\gamma-x)(x-\delta)}}$$

$$= \frac{2}{\sqrt{(a-\gamma)(\beta-\delta)}} \operatorname{sn}^{-1}\left(\sqrt{\frac{\beta-\delta}{\gamma-\delta}\cdot\frac{\gamma-x}{\beta-x}},\ \sqrt{\frac{a-\beta}{a-\gamma}\cdot\frac{\gamma-\delta}{\beta-\delta}}\right).$$

$\gamma > x > \delta.$

556. $$\int_\delta^x \frac{dx}{\sqrt{(a-x)(\beta-x)(\gamma-x)(x-\delta)}}$$

$$= \frac{2}{\sqrt{(a-\gamma)(\beta-\delta)}} \operatorname{sn}^{-1}\left(\sqrt{\frac{a-\gamma}{\gamma-\delta}\cdot\frac{x-\delta}{a-x}},\ \sqrt{\frac{a-\beta}{a-\gamma}\cdot\frac{\gamma-\delta}{\beta-\delta}}\right).$$

$\gamma > x > \delta.$

557. $$\int_x^\delta \frac{dx}{\sqrt{(a-x)(\beta-x)(\gamma-x)(\delta-x)}}$$

$$= \frac{2}{\sqrt{(a-\gamma)(\beta-\delta)}} \operatorname{sn}^{-1}\left(\sqrt{\frac{a-\gamma}{a-\delta}\cdot\frac{\delta-x}{\gamma-x}},\ \sqrt{\frac{\beta-\gamma}{a-\gamma}\cdot\frac{a-\delta}{\beta-\delta}}\right).$$

$\delta > x.$

558. $$\int \operatorname{sn} x\, dx = -\frac{1}{k} \cosh^{-1}\left(\frac{\operatorname{dn} x}{k'}\right).$$

559. $$\int \operatorname{cn} x\, dx = \frac{1}{k} \cos^{-1}(\operatorname{dn} x).$$

560. $\int \mathrm{dn}\, x\, dx = \sin^{-1}(\mathrm{sn}\, x) = \mathrm{am}\, x.$

561. $\int \frac{dx}{\mathrm{sn}\, x} = \log\left[\frac{\mathrm{sn}\, x}{\mathrm{cn}\, x + \mathrm{dn}\, x}\right].$

562. $\int \frac{dx}{\mathrm{cn}\, x} = \frac{1}{k'} \log\left[\frac{k'\, \mathrm{sn}\, x + \mathrm{dn}\, x}{\mathrm{cn}\, x}\right].$

563. $\int \frac{dx}{\mathrm{dn}\, x} = \frac{1}{k'} \tan^{-1}\left[\frac{k'\, \mathrm{sn}\, x - \mathrm{cn}\, x}{k'\, \mathrm{sn}\, x + \mathrm{cn}\, x}\right].$

564. $\int_0^x \mathrm{sn}^2 x\, dx = \frac{1}{k^2}[x - E(\mathrm{am}\, x,\ k)].$

565. $\int_0^x \mathrm{cn}^2 x\, dx = \frac{1}{k^2}[E(\mathrm{am}\, x,\ k) - k'^2 x].$

566. $\int_0^x \mathrm{dn}^2 x\, dx = E(\mathrm{am}\, x,\ k).$

567. $(m+1)\int \mathrm{sn}^m x\, dx = (m+2)(1+k^2)\int \mathrm{sn}^{m+2} x\, dx$
$$-(m+3)k^2 \int \mathrm{sn}^{m+4} x\, dx + \mathrm{sn}^{m+1} x\ \mathrm{cn}\, x\ \mathrm{dn}\, x.$$

568. $(m+1)k'^2 \int \mathrm{cn}^m x\, dx = (m+2)(1-2k^2)\int \mathrm{cn}^{m+2} x\, dx$
$$+(m+3)k^2 \int \mathrm{cn}^{m+4} x\, dx - \mathrm{cn}^{m+1} x\ \mathrm{sn}\, x\ \mathrm{dn}\, x.$$

569. $(m+1)k'^2 \int \mathrm{dn}^m x\, dx = (m+2)(2-k^2)\int \mathrm{dn}^{m+2} x\, dx$
$$-(m+3)\int \mathrm{dn}^{m+4} x\, dx + k^2\, \mathrm{dn}^{m+1} x\, \mathrm{sn}\, x\, \mathrm{cn}\, x.$$

Since $\sin^2\theta \equiv \frac{1}{k^2} - \frac{1}{k^2}(1 - k^2 \cdot \sin^2\theta)$,

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2\theta \cdot d\theta}{\sqrt{1-k^2\sin^2\theta}} = \frac{1}{k^2}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} - \frac{1}{k^2}\int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\theta}\cdot d\theta.$$

## VIII. AUXILIARY FORMULAS.

### A. — Trigonometric Functions.

**570.** $\tan a \cdot \operatorname{ctn} a = \sin a \cdot \csc a = \cos a \cdot \sec a = 1,$

$\tan a = \sin a \div \cos a, \quad \sec^2 a = 1 + \tan^2 a,$

$\csc^2 a = 1 + \operatorname{ctn}^2 a, \quad \sin^2 a + \cos^2 a = 1.$

**571.** $\sin a = \sqrt{1 - \cos^2 a} = 2 \sin \tfrac{1}{2} a \cdot \cos \tfrac{1}{2} a = \cos a \cdot \tan a$

$$= \frac{1}{\sqrt{1 + \operatorname{ctn}^2 a}} = \frac{\tan a}{\sqrt{1 + \tan^2 a}} = \sqrt{\frac{1 - \cos 2a}{2}} = \frac{2 \tan \tfrac{1}{2} a}{1 + \tan^2 \tfrac{1}{2} a}$$

$$= \sqrt{\frac{\sec^2 a - 1}{\sec^2 a}} = \operatorname{ctn} \tfrac{1}{2} a \cdot (1 - \cos a) = \tan \tfrac{1}{2} a \cdot (1 + \cos a).$$

**572.** $\cos a = \sqrt{1 - \sin^2 a} = \frac{1}{\sqrt{1 + \tan^2 a}} = \frac{\operatorname{ctn} a}{\sqrt{1 + \operatorname{ctn}^2 a}}$

$$= \sqrt{\frac{1 + \cos 2a}{2}} = \frac{1 - \tan^2 \tfrac{1}{2} a}{1 + \tan^2 \tfrac{1}{2} a} = \cos^2 \tfrac{1}{2} a - \sin^2 \tfrac{1}{2} a$$

$$= 1 - 2 \sin^2 \tfrac{1}{2} a = 2 \cos^2 \tfrac{1}{2} a - 1 = \sin a \cdot \operatorname{ctn} a$$

$$= \frac{\sin 2a}{2 \sin a} = \sqrt{\frac{\csc^2 a - 1}{\csc^2 a}} = \frac{\operatorname{ctn} \tfrac{1}{2} a - \tan \tfrac{1}{2} a}{\operatorname{ctn} \tfrac{1}{2} a + \tan \tfrac{1}{2} a}.$$

**573.** $\tan a = \frac{\sin a}{\sqrt{1 - \sin^2 a}} = \frac{\sqrt{1 - \cos^2 a}}{\cos a} = \frac{\sin 2a}{1 + \cos 2a}$

$$= \frac{1 - \cos 2a}{\sin 2a} = \sqrt{\frac{1 - \cos 2a}{1 + \cos 2a}} = \frac{2 \tan \tfrac{1}{2} a}{1 - \tan^2 \tfrac{1}{2} a}$$

$$= \frac{\sec a}{\csc a} = \frac{2}{\operatorname{ctn} \tfrac{1}{2} a - \tan \tfrac{1}{2} a} = \frac{2 \operatorname{ctn} \tfrac{1}{2} a}{\operatorname{ctn}^2 \tfrac{1}{2} a - 1}.$$

**574.**

| | $-\alpha$. | $90° \pm \alpha$. | $180° \pm \alpha$. | $270° \pm \alpha$. | $360° \pm \alpha$. |
|---|---|---|---|---|---|
| sin | $-\sin\alpha$ | $+\cos\alpha$ | $\mp\sin\alpha$ | $-\cos\alpha$ | $\pm\sin\alpha$ |
| cos | $+\cos\alpha$ | $\mp\sin\alpha$ | $-\cos\alpha$ | $\pm\sin\alpha$ | $+\cos\alpha$ |
| tan | $-\tan\alpha$ | $\mp\operatorname{ctn}\alpha$ | $\pm\tan\alpha$ | $\mp\operatorname{ctn}\alpha$ | $\pm\tan\alpha$ |
| ctn | $-\operatorname{ctn}\alpha$ | $\mp\tan\alpha$ | $\pm\operatorname{ctn}\alpha$ | $\mp\tan\alpha$ | $\pm\operatorname{ctn}\alpha$ |
| sec | $+\sec\alpha$ | $\mp\csc\alpha$ | $-\sec\alpha$ | $\pm\csc\alpha$ | $+\sec\alpha$ |
| csc | $-\csc\alpha$ | $+\sec\alpha$ | $\mp\csc\alpha$ | $-\sec\alpha$ | $\pm\csc\alpha$ |

**575.**

| | 0°. | 30°. | 45°. | 60°. | 90°. | 120°. | 135°. | 150°. | 180°. |
|---|---|---|---|---|---|---|---|---|---|
| sin | 0 | $\frac{1}{2}$ | $\frac{1}{2}\sqrt{2}$ | $\frac{1}{2}\sqrt{3}$ | 1 | $\frac{1}{2}\sqrt{3}$ | $\frac{1}{2}\sqrt{2}$ | $\frac{1}{2}$ | 0 |
| cos | 1 | $\frac{1}{2}\sqrt{3}$ | $\frac{1}{2}\sqrt{2}$ | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $-\frac{1}{2}\sqrt{2}$ | $-\frac{1}{2}\sqrt{3}$ | $-1$ |
| tan | 0 | $\frac{1}{\sqrt{3}}$ | 1 | $\sqrt{3}$ | $\infty$ | $-\sqrt{3}$ | $-1$ | $-\frac{1}{\sqrt{3}}$ | 0 |
| ctn | $\infty$ | $\sqrt{3}$ | 1 | $\frac{1}{\sqrt{3}}$ | 0 | $-\frac{1}{\sqrt{3}}$ | $-1$ | $-\sqrt{3}$ | $\infty$ |
| sec | 1 | $\frac{2}{\sqrt{3}}$ | $\sqrt{2}$ | 2 | $\infty$ | $-2$ | $-\sqrt{2}$ | $-\frac{2}{\sqrt{3}}$ | $-1$ |
| csc | $\infty$ | 2 | $\sqrt{2}$ | $\frac{2}{\sqrt{3}}$ | 1 | $\frac{2}{\sqrt{3}}$ | $\sqrt{2}$ | 2 | $\infty$ |

**576.** $\sin\frac{1}{2}a = \sqrt{\frac{1}{2}(1-\cos a)}$.

**577.** $\cos\frac{1}{2}a = \sqrt{\frac{1}{2}(1+\cos a)}$.

**578.** $\tan\frac{1}{2}a = \sqrt{\frac{1-\cos a}{1+\cos a}} = \frac{1-\cos a}{\sin a} = \frac{\sin a}{1+\cos a}$.

**579.** $\sin 2a = 2\sin a\cos a$.

**580.** $\sin 3a = 3\sin a - 4\sin^3 a$.

**581.** $\sin 4a = 8\cos^3 a\cdot\sin a - 4\cos a\sin a$.

582. $\sin 5\,a = 5 \sin a - 20 \sin^3 a + 16 \sin^5 a.$

583. $\sin 6\,a = 32 \cos^5 a \sin a - 32 \cos^3 a \sin a + 6 \cos a \sin a.$

584. $\cos 2\,a = \cos^2 a - \sin^2 a = 1 - 2 \sin^2 a = 2 \cos^2 a - 1.$

585. $\cos 3\,a = 4 \cos^3 a - 3 \cos a.$

586. $\cos 4\,a = 8 \cos^4 a - 8 \cos^2 a + 1.$

587. $\cos 5\,a = 16 \cos^5 a - 20 \cos^3 a + 5 \cos a.$

588. $\cos 6\,a = 32 \cos^6 a - 48 \cos^4 a + 18 \cos^2 a - 1.$

589. $\tan 2\,a = \dfrac{2 \tan a}{1 - \tan^2 a}.$

590. $\operatorname{ctn} 2\,a = \dfrac{\operatorname{ctn}^2 a - 1}{2 \operatorname{ctn} a}.$

591. $\sin(a \pm \beta) = \sin a \cdot \cos \beta \pm \cos x \cdot \sin \beta.$

592. $\cos(a \pm \beta) = \cos a \cdot \cos \beta \mp \sin a \cdot \sin \beta.$

593. $\tan(a \pm \beta) = \dfrac{\tan a \pm \tan \beta}{1 \mp \tan a \cdot \tan \beta}.$

594. $\operatorname{ctn}(a \pm \beta) = \dfrac{\operatorname{ctn} a \cdot \operatorname{ctn} \beta \mp 1}{\operatorname{ctn} \beta \pm \operatorname{ctn} a}.$

595. $\sin a \pm \sin \beta = 2 \sin \tfrac{1}{2}(a \pm \beta) \cdot \cos \tfrac{1}{2}(a \mp \beta).$

596. $\cos a + \cos \beta = 2 \cos \tfrac{1}{2}(a + \beta) \cdot \cos \tfrac{1}{2}(a - \beta).$

597. $\cos a - \cos \beta = -2 \sin \tfrac{1}{2}(a + \beta) \cdot \sin \tfrac{1}{2}(a - \beta).$

598. $\tan a \pm \tan \beta = \dfrac{\sin(a \pm \beta)}{\cos a \cdot \cos \beta}.$

599. $\operatorname{ctn} a \pm \operatorname{ctn} \beta = \pm \dfrac{\sin(a \pm \beta)}{\sin a \cdot \sin \beta}.$

600. $\dfrac{\sin \alpha \pm \sin \beta}{\cos \alpha + \cos \beta} = \tan \tfrac{1}{2}(\alpha \pm \beta).$

601. $\dfrac{\sin \alpha \pm \sin \beta}{\cos \alpha - \cos \beta} = -\operatorname{ctn} \tfrac{1}{2}(\alpha \mp \beta).$

602. $\dfrac{\sin \alpha + \sin \beta}{\sin \alpha - \sin \beta} = \dfrac{\tan \tfrac{1}{2}(\alpha + \beta)}{\tan \tfrac{1}{2}(\alpha - \beta)}.$

603. $\sin^2 \alpha - \sin^2 \beta = \sin(\alpha + \beta) \cdot \sin(\alpha - \beta).$

604. $\cos^2 \alpha - \cos^2 \beta = -\sin(\alpha + \beta) \cdot \sin(\alpha - \beta).$

605. $\cos^2 \alpha - \sin^2 \beta = \cos(\alpha + \beta) \cdot \cos(\alpha - \beta).$

606. $\sin xi = \tfrac{1}{2} i(e^x - e^{-x}) = i \sinh x.$

607. $\cos xi = \tfrac{1}{2}(e^x + e^{-x}) = \cosh x.$

608. $\tan xi = \dfrac{i(e^x - e^{-x})}{e^x + e^{-x}} = i \tanh x.$

609. $e^{x+yi} = e^x \cos y + ie^x \sin y.$

610. $a^{x+yi} = a^x \cos(y \cdot \log a) + ia^x \sin(y \cdot \log a).$

611. $(\cos \theta \pm i \cdot \sin \theta)^n = \cos n\theta \pm i \cdot \sin n\theta.$

612. $\sin x = -\tfrac{1}{2} i(e^{xi} - e^{-xi}).$

613. $\cos x = \tfrac{1}{2}(e^{xi} + e^{-xi}).$

614. $\tan x = -i \dfrac{e^{2xi} - 1}{e^{2xi} + 1}.$

615. $\sin(x \pm yi) = \sin x \cos yi \pm \cos x \sin yi$
$= \sin x \cosh y \pm i \cos x \sinh y.$

616. $\cos(x \pm yi) = \cos x \cos yi \mp \sin x \sin yi$
$= \cos x \cosh y \mp i \sin x \sinh y.$

In any plane triangle,

**617.** $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

**618.** $a^2 = b^2 + c^2 - 2bc \cos A$.

**619.** $\dfrac{a+b}{a-b} = \dfrac{\sin A + \sin B}{\sin A - \sin B} = \dfrac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)} = \dfrac{\operatorname{ctn} \frac{1}{2} C}{\tan \frac{1}{2}(A-B)}$.

**620.** $\sin \frac{1}{2} A = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$, where $2s = a + b + c$.

**621.** $\cos \frac{1}{2} A = \sqrt{\dfrac{s(s-a)}{bc}}$.

**622.** $\tan \frac{1}{2} A = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$.

**623.** Area $= \frac{1}{2} bc \sin A = \sqrt{s(s-a)(s-b)(s-c)}$.

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In any spherical triangle,

**624.** $\dfrac{\sin A}{\sin a} = \dfrac{\sin B}{\sin b} = \dfrac{\sin C}{\sin c}$.

**625.** $\cos a = \cos b \cos c + \sin b \sin c \cos A$.

**626.** $-\cos A = \cos B \cos C - \sin B \sin C \cos a$.

**627.** $\sin a \operatorname{ctn} b = \sin C \operatorname{ctn} B + \cos a \cos C$.

**628.** $\cos \frac{1}{2} A = \sqrt{\dfrac{\sin s \cdot \sin (s-a)}{\sin b \cdot \sin c}}$.

**629.** $\sin \frac{1}{2} A = \sqrt{\dfrac{\sin (s-b) \cdot \sin (s-c)}{\sin b \cdot \sin c}}$.

**630.** $\tan \frac{1}{2} A = \sqrt{\dfrac{\sin (s-b) \cdot \sin (s-c)}{\sin s \cdot \sin (s-a)}}$.

631. $\cos \frac{1}{2} a = \sqrt{\frac{\cos (S-B) \cdot \cos (S-C)}{\sin B \cdot \sin C}}$.

632. $\sin \frac{1}{2} a = \sqrt{\frac{-\cos S \cdot \cos (S-A)}{\sin B \sin C}}$.

633. $\tan \frac{1}{2} a = \sqrt{\frac{-\cos S \cdot \cos (S-A)}{\cos (S-B) \cdot \cos (S-C)}}$.

$2s = a + b + c. \qquad 2S = A + B + C.$

634. $\cos \frac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(a+b)}{\cos \frac{1}{2} c} \sin \frac{1}{2} C.$

635. $\cos \frac{1}{2}(A-B) = \frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2} c} \sin \frac{1}{2} C.$

636. $\sin \frac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2} c} \cos \frac{1}{2} C.$

637. $\sin \frac{1}{2}(A-B) = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2} c} \cos \frac{1}{2} C.$

638. $\tan \frac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \operatorname{ctn} \frac{1}{2} C.$

639. $\tan \frac{1}{2}(A-B) = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \operatorname{ctn} \frac{1}{2} C.$

640. $\tan \frac{1}{2}(a+b) = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} \tan \frac{1}{2} c.$

641. $\tan \frac{1}{2}(a-b) = \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} \tan \frac{1}{2} c.$

642. $\frac{\cos \frac{1}{2}(a+b)}{\cos \frac{1}{2}(a-b)} = \frac{\operatorname{ctn} \frac{1}{2} C}{\tan \frac{1}{2}(A+B)}$.

In interpreting equations which involve logarithmic and anti-trigonometric functions, it is necessary to remember that these functions are multiple valued. To save space the formulas on this page and the next are printed in contracted form.

**643.** $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}} = \sec^{-1}\dfrac{1}{\sqrt{1-x^2}}$

$= \csc^{-1}\dfrac{1}{x} = 2\sin^{-1}[\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1-x^2}]^{\frac{1}{2}}$

$= \tfrac{1}{2}\sin^{-1}(2x\sqrt{1-x^2}) = 2\tan^{-1}\left[\dfrac{1-\sqrt{1-x^2}}{x}\right]$

$= \tfrac{1}{2}\tan^{-1}\left[\dfrac{2x\sqrt{1-x^2}}{1-2x^2}\right] = \tfrac{1}{2}\pi - \cos^{-1}x$

$= \tfrac{1}{2}\pi - \sin^{-1}\sqrt{1-x^2} = -\sin^{-1}(-x)$

$= \operatorname{ctn}^{-1}\dfrac{\sqrt{1-x^2}}{x} = (2n+\tfrac{1}{2})\pi - i\log(x+\sqrt{x^2-1})$

$= \tfrac{1}{4}\pi + \tfrac{1}{2}\sin^{-1}(2x^2-1) = \tfrac{1}{2}\cos^{-1}(1-2x^2).$

**644.** $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\dfrac{\sqrt{1-x^2}}{x} = \sec^{-1}\dfrac{1}{x}$

$= \tfrac{1}{2}\pi - \sin^{-1}x = 2\cos^{-1}\sqrt{\dfrac{1+x}{2}}$

$= \tfrac{1}{2}\cos^{-1}(2x^2-1)$

$= 2\tan^{-1}\sqrt{\dfrac{1-x}{1+x}} = \tfrac{1}{2}\tan^{-1}\left[\dfrac{2x\sqrt{1-x^2}}{2x^2-1}\right]$

$= \csc^{-1}\dfrac{1}{\sqrt{1-x^2}} = \pi - \cos^{-1}(-x)$

$= \operatorname{ctn}^{-1}\dfrac{x}{\sqrt{1-x^2}}$

$= i\log(x+\sqrt{x^2-1}) = \pi - i\log(\sqrt{x^2-1}-x).$

**645.** $\tan^{-1}x = \sin^{-1}\dfrac{x}{\sqrt{1+x^2}} = \cos^{-1}\dfrac{1}{\sqrt{1+x^2}} = \frac{1}{2}\sin^{-1}\dfrac{2x}{1+x^2}$

$= \operatorname{ctn}^{-1}\dfrac{1}{x} = \frac{1}{2}\pi - \operatorname{ctn}^{-1}x = \sec^{-1}\sqrt{1+x^2}$

$= \frac{1}{2}\pi - \tan^{-1}\dfrac{1}{x}$

$= \csc^{-1}\dfrac{\sqrt{1+x^2}}{x} = \frac{1}{2}\cos^{-1}\left[\dfrac{1-x^2}{1+x^2}\right]$

$= 2\cos^{-1}\left[\dfrac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}\right]^{\frac{1}{2}} = 2\sin^{-1}\left[\dfrac{\sqrt{1+x^2}-1}{2\sqrt{1+x^2}}\right]^{\frac{1}{2}}$

$= \frac{1}{2}\tan^{-1}\dfrac{2x}{1-x^2} = 2\tan^{-1}\left[\dfrac{\sqrt{1+x^2}-1}{x}\right]$

$= -\tan^{-1}c + \tan^{-1}\left[\dfrac{x+c}{1-cx}\right] = -\tan^{-1}(-x)$

$= \frac{1}{2}i\log\dfrac{1-xi}{1+xi} = \frac{1}{2}i\log\dfrac{i+x}{i-x}$

$= -\frac{1}{2}i\log\dfrac{1+xi}{1-xi}.$

**646.** $\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}[x\sqrt{1-y^2} \pm y\sqrt{1-x^2}].$

**647.** $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}[xy \mp \sqrt{(1-x^2)(1-y^2)}].$

**648.** $\tan^{-1}x \pm \tan^{-1}y = \tan^{-1}\left[\dfrac{x \pm y}{1 \mp xy}\right].$

**649.** $\sin^{-1}x \pm \cos^{-1}y = \sin^{-1}[xy \pm \sqrt{(1-x^2)(1-y^2)}]$

$= \cos^{-1}[y\sqrt{1-x^2} \mp x\sqrt{1-y^2}].$

**650.** $\tan^{-1}x \pm \operatorname{ctn}^{-1}y = \tan^{-1}\left[\dfrac{xy \pm 1}{y \mp x}\right] = \operatorname{ctn}^{-1}\left[\dfrac{y \mp x}{xy \pm 1}\right]$

**651.** $\log(x+yi) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x).$

## B. — Hyperbolic Functions.

**652.** $\sinh x = \frac{1}{2}(e^x - e^{-x}) = -\sinh(-x) = -i\sin(ix)$
$= (\operatorname{csch} x)^{-1} = 2\tanh\frac{1}{2}x \div (1 - \tanh^2\frac{1}{2}x).$

**653.** $\cosh x = \frac{1}{2}(e^x + e^{-x}) = \cosh(-x) = \cos(ix) = (\operatorname{sech} x)^{-1}$
$= (1 + \tanh^2\frac{1}{2}x) \div (1 - \tanh^2\frac{1}{2}x).$

**654.** $\tanh x = (e^x - e^{-x}) \div (e^x + e^{-x}) = -\tanh(-x)$
$= -i\tan(ix) = (\operatorname{ctnh} x)^{-1} = \sinh x \div \cosh x.$

**655.** $\cosh xi = \cos x.$

**656.** $\sinh xi = i\sin x.$

**657.** $\cosh^2 x - \sinh^2 x = 1.$

**658.** $1 - \tanh^2 x = \operatorname{sech}^2 x.$

**659.** $1 - \operatorname{ctnh}^2 x = -\operatorname{csch}^2 x.$

**660.** $\sinh(x \pm y) = \sinh x \cdot \cosh y \pm \cosh x \cdot \sinh y.$

**661.** $\cosh(x \pm y) = \cosh x \cdot \cosh y \pm \sinh x \cdot \sinh y.$

**662.** $\tanh(x \pm y) = (\tanh x \pm \tanh y) \div (1 \pm \tanh x \cdot \tanh y).$

**663.** $\sinh(2x) = 2\sinh x \cosh x.$

**664.** $\cosh(2x) = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x.$

**665.** $\tanh(2x) = 2\tanh x \div (1 + \tanh^2 x).$

**666.** $\sinh(\frac{1}{2}x) = \sqrt{\frac{1}{2}(\cosh x - 1)}.$

**667.** $\cosh(\frac{1}{2}x) = \sqrt{\frac{1}{2}(\cosh x + 1)}.$

**668.** $\tanh(\frac{1}{2}x) = (\cosh x - 1) \div \sinh x = \sinh x \div (\cosh x + 1).$

**669.** $\sinh x + \sinh y = 2\sinh\frac{1}{2}(x + y) \cdot \cosh\frac{1}{2}(x - y).$

**670.** $\sinh x - \sinh y = 2\cosh\frac{1}{2}(x + y) \cdot \sinh\frac{1}{2}(x - y).$

**671.** $\cosh x + \cosh y = 2 \cosh \tfrac{1}{2}(x+y) \cdot \cosh \tfrac{1}{2}(x-y).$

**672.** $\cosh x - \cosh y = 2 \sinh \tfrac{1}{2}(x+y) \cdot \sinh \tfrac{1}{2}(x-y).$

**673.** $d \sinh x = \cosh x \cdot dx.$

**674.** $d \cosh x = \sinh x \cdot dx.$

**675.** $d \tanh x = \operatorname{sech}^2 x \cdot dx.$

**676.** $d \operatorname{ctnh} x = -\operatorname{csch}^2 x \cdot dx.$

**677.** $d \operatorname{sech} x = -\operatorname{sech} x \cdot \tanh x \cdot dx.$

**678.** $d \operatorname{csch} x = -\operatorname{csch} x \cdot \operatorname{ctnh} x \cdot dx.$

**679.** $\sinh^{-1} x = \log(x + \sqrt{x^2+1}) = \int \frac{dx}{\sqrt{x^2+1}}$
$= \cosh^{-1} \sqrt{x^2+1}.$

**680.** $\cosh^{-1} x = \log(x + \sqrt{x^2-1}) = \int \frac{dx}{\sqrt{x^2-1}}$
$= \sinh^{-1} \sqrt{x^2-1}.$

**681.** $\tanh^{-1} x = \tfrac{1}{2} \log(1+x) - \tfrac{1}{2} \log(1-x) = \int \frac{dx}{1-x^2}.$

**682.** $\operatorname{ctnh}^{-1} x = \tfrac{1}{2} \log(1+x) - \tfrac{1}{2} \log(x-1) = \int \frac{dx}{1-x^2}.$

**683.** $\operatorname{sech}^{-1} x = \log\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1}\right) = -\int \frac{dx}{x\sqrt{1-x^2}}.$

**684.** $\operatorname{csch}^{-1} x = \log\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1}\right) = -\int \frac{dx}{x\sqrt{x^2+1}}.$

**685.** $d \sinh^{-1} x = \frac{dx}{\sqrt{1+x^2}}.$

**686.** $d \cosh^{-1} x = \frac{dx}{\sqrt{x^2-1}}.$

**687.** $d \tanh^{-1} x = \dfrac{dx}{1 - x^2}$.

**688.** $d \operatorname{ctnh}^{-1} x = -\dfrac{dx}{x^2 - 1}$.

**689.** $d \operatorname{sech}^{-1} x = -\dfrac{dx}{x\sqrt{1 - x^2}}$.

**690.** $d \operatorname{csch}^{-1} x = -\dfrac{dx}{x\sqrt{x^2 + 1}}$.

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If $m$ is an integer,

**691.** $\sinh(m\pi i) = 0$.

**692.** $\cosh(m\pi i) = \cos m\pi = (-1)^m$.

**693.** $\tanh(m\pi i) = 0$.

**694.** $\sinh(x + m\pi i) = (-1)^m \sinh x$.

**695.** $\cosh(x + m\pi i) = (-1)^m \cosh(x)$.

**696.** $\sinh(2m + 1)\frac{1}{2}\pi i = i \sin(2m + 1)\frac{1}{2}\pi = \pm i$.

**697.** $\cosh(2m + 1)\frac{1}{2}\pi i = 0$.

**698.** $\sinh\left(\dfrac{\pi i}{2} \pm x\right) = i \cosh x$.

**799.** $\cosh\left(\dfrac{\pi i}{2} \pm x\right) = \pm i \sinh x$.

**700.** $\sinh u = \tan \operatorname{gd} u$.

**701.** $\cosh u = \sec \operatorname{gd} u$.

**702.** $\tanh u = \sin \operatorname{gd} u$.

**703.** $\tanh \frac{1}{2} u = \tan \frac{1}{2} \operatorname{gd} u$.

**704.** $u = \log \tan(\frac{1}{4}\pi + \frac{1}{2} \operatorname{gd} u)$. $\int \sec x\, dx = gd^{-1} x$.

## C. — Elliptic Functions.

If $u = F(\phi, k) \equiv \int_0^x \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} \equiv \int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$,

where $k < 1$, and $x \equiv \sin\phi$, $\phi$ is called the *amplitude* of $u$ and is written $\operatorname{am}(u, \operatorname{mod} k)$, or, more simply, $\operatorname{am} u$; $x \equiv \sin\phi \equiv \operatorname{sn} u$,

$$\sqrt{1-x^2} \equiv \cos\phi \equiv \operatorname{cn} u, \quad \sqrt{1-k^2x^2} \equiv \Delta\phi \equiv \operatorname{dn} u \equiv \operatorname{dn} u,$$

$$K \equiv F(\tfrac{1}{2}\pi, k), \qquad K' \equiv F(\tfrac{1}{2}\pi, k').$$

Hence, $\operatorname{am}(0) = 0$, $\operatorname{sn}(0) = 0$, $\operatorname{cn}(0) = 1$, $\operatorname{dn}(0) = 1$,

$$\operatorname{am}(-u) = -\operatorname{am} u, \qquad \operatorname{sn}(-u) = -\operatorname{sn} u,$$

$$\operatorname{cn}(-u) = \operatorname{cn} u, \qquad \operatorname{dn}(-u) = \operatorname{dn} u.$$

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**705.** $\operatorname{sn}^2 u + \operatorname{cn}^2 u = 1.$

**706.** $\operatorname{dn}^2 u + k^2 \operatorname{sn}^2 u = 1.$

**707.** $\operatorname{dn}^2 u - k^2 \operatorname{cn}^2 u = 1 - k^2 = k'^2.$

**708.** $\operatorname{sn} 2u = \dfrac{2 \operatorname{sn} u \cdot \operatorname{cn} u \cdot \operatorname{dn} u}{1 - k^2 \operatorname{sn}^4 u}.$

**709.** $\operatorname{cn} 2u = \dfrac{\operatorname{cn}^2 u - \operatorname{sn}^2 u \cdot \operatorname{dn}^2 u}{1 - k^2 \operatorname{sn}^4 u} = \dfrac{1 - 2\operatorname{sn}^2 u + k^2 \operatorname{sn}^4 u}{1 - k^2 \operatorname{sn}^4 u}$

$$= 1 - \frac{2 \operatorname{sn}^2 u \cdot \operatorname{dn}^2 u}{1 - k^2 \operatorname{sn}^4 u} = \frac{2 \operatorname{cn}^2 u}{1 - k^2 \operatorname{sn}^4 u} - 1.$$

**710.** $\operatorname{dn} 2u = \dfrac{\operatorname{dn}^2 u - k^2 \operatorname{sn}^2 u \cdot \operatorname{cn}^2 u}{1 - k^2 \operatorname{sn}^4 u} = \dfrac{1 - 2k^2 \operatorname{sn}^2 u + k^2 \operatorname{sn}^4 u}{1 - k^2 \operatorname{sn}^4 u}$

$$= 1 - \frac{2k^2 \operatorname{sn}^2 u \cdot \operatorname{cn}^2 u}{1 - k^2 \operatorname{sn}^4 u} = \frac{2 \operatorname{dn}^2 u}{1 - k^2 \operatorname{sn}^4 u} - 1.$$

**711.** $\operatorname{sn}^2\left(\dfrac{u}{2}\right) = \dfrac{1 - \operatorname{cn} u}{1 + \operatorname{dn} u} = \dfrac{1 - \operatorname{dn} u}{k^2(1 + \operatorname{cn} u)} = \dfrac{\operatorname{dn} u - \operatorname{cn} u}{k'^2 + \operatorname{dn} u - k^2 \operatorname{cn} u}.$

**712.** $\operatorname{cn}^2\left(\dfrac{u}{2}\right) = \dfrac{\operatorname{dn} u + \operatorname{cn} u}{1 + \operatorname{dn} u} = \dfrac{k^2 \operatorname{cn} u - k'^2 + \operatorname{dn} u}{k^2(1 + \operatorname{cn} u)}$

$$= \frac{k'^2(1 + \operatorname{cn} u)}{k'^2 + \operatorname{dn} u - k^2 \operatorname{cn} u}.$$

**713.** $$\mathrm{dn}^2\left(\frac{u}{2}\right) = \frac{k'^2 + \mathrm{dn}\,u + k^2\,\mathrm{cn}\,u}{1 + \mathrm{dn}\,u} = \frac{k^2(\mathrm{cn}\,u + \mathrm{dn}\,u)}{k^2(1 + \mathrm{cn}\,u)}$$

$$= \frac{k'^2(1 + \mathrm{dn}\,u)}{k'^2 + \mathrm{dn}\,u - k^2\,\mathrm{cn}\,u}.$$

If, moreover, $v = \int_0^y \frac{dz}{\sqrt{(1 - z^2)(1 - k^2z^2)}}$,

**714.** $$\mathrm{sn}^2 u - \mathrm{sn}^2 v = \mathrm{cn}^2 v - \mathrm{cn}^2 u.$$

**715.** $$\mathrm{sn}(u \pm v) = \frac{\mathrm{sn}\,u \cdot \mathrm{cn}\,v \cdot \mathrm{dn}\,v \pm \mathrm{cn}\,u \cdot \mathrm{sn}\,v \cdot \mathrm{dn}\,u}{1 - k^2\,\mathrm{sn}^2 u \cdot \mathrm{sn}^2 v}.$$

**716.** $$\mathrm{cn}(u \pm v) = \frac{\mathrm{cn}\,u \cdot \mathrm{cn}\,v \mp \mathrm{sn}\,u \cdot \mathrm{sn}\,v \cdot \mathrm{dn}\,u \cdot \mathrm{dn}\,v}{1 - k^2\,\mathrm{sn}^2 u \cdot \mathrm{sn}^2 v}$$

$$= \mathrm{cn}\,u \cdot \mathrm{cn}\,v \mp \mathrm{sn}\,u \cdot \mathrm{sn}\,v \cdot \mathrm{dn}(u \pm v).$$

**717.** $$\mathrm{dn}(u \pm v) = \frac{\mathrm{dn}\,u \cdot \mathrm{dn}\,v \mp k^2\,\mathrm{sn}\,u \cdot \mathrm{sn}\,v \cdot \mathrm{cn}\,u \cdot \mathrm{cn}\,v}{1 - k^2\,\mathrm{sn}^2 u \cdot \mathrm{sn}^2 v}$$

$$= \mathrm{dn}\,u \cdot \mathrm{dn}\,v \mp k^2\,\mathrm{sn}\,u \cdot \mathrm{sn}\,v \cdot \mathrm{cn}(u \pm v).$$

**718.** $$\mathrm{tn}(u \pm v) = \frac{\mathrm{tn}\,u \cdot \mathrm{dn}\,v \pm \mathrm{tn}\,v \cdot \mathrm{dn}\,u}{1 \mp \mathrm{tn}\,u \cdot \mathrm{tn}\,v \cdot \mathrm{dn}\,u \cdot \mathrm{dn}\,v}.$$

**719.** $$\mathrm{sn}(u + v) + \mathrm{sn}(u - v) = \frac{2\,\mathrm{sn}\,u \cdot \mathrm{cn}\,v \cdot \mathrm{dn}\,v}{1 - k^2\,\mathrm{sn}^2 u \cdot \mathrm{sn}^2 v}.$$

**720.** $$\mathrm{sn}(u + v) - \mathrm{sn}(u - v) = \frac{2\,\mathrm{sn}\,v \cdot \mathrm{cn}\,u \cdot \mathrm{dn}\,u}{1 - k^2\,\mathrm{sn}^2 u \cdot \mathrm{sn}^2 v}.$$

**721.** $$\mathrm{cn}(u + v) + \mathrm{cn}(u - v) = \frac{2\,\mathrm{cn}\,u \cdot \mathrm{cn}\,v}{1 - k^2\,\mathrm{sn}^2 u \cdot \mathrm{sn}^2 v}.$$

**722.** $$\mathrm{cn}(u + v) - \mathrm{cn}(u - v) = -\frac{2\,\mathrm{sn}\,u \cdot \mathrm{sn}\,v \cdot \mathrm{dn}\,u \cdot \mathrm{dn}\,v}{1 - k^2\,\mathrm{sn}^2 u \cdot \mathrm{sn}^2 v}.$$

**723.** $$\mathrm{dn}(u + v) + \mathrm{dn}(u - v) = \frac{2\,\mathrm{dn}\,u \cdot \mathrm{dn}\,v}{1 - k^2\,\mathrm{sn}^2 u \cdot \mathrm{sn}^2 v}.$$

**724.** $\operatorname{dn}(u+v) - \operatorname{dn}(u-v) = -\dfrac{2k^2 \operatorname{sn} u \cdot \operatorname{sn} v \cdot \operatorname{cn} u \cdot \operatorname{cn} v}{1 - k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v}.$

**725.** $\operatorname{sn}(u+v) \cdot \operatorname{sn}(u-v) = \dfrac{\operatorname{sn}^2 u - \operatorname{sn}^2 v}{1 - k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v}$

$= \dfrac{\operatorname{cn}^2 v + \operatorname{sn}^2 u \cdot \operatorname{dn}^2 v}{1 - k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v} - 1 = \dfrac{1}{k^2}\left[\dfrac{\operatorname{dn}^2 v + k^2 \operatorname{sn}^2 u \cdot \operatorname{cn}^2 v}{1 - k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v} - 1\right].$

**726.** $\operatorname{cn}(u+v) \cdot \operatorname{cn}(u-v) = \dfrac{\operatorname{cn}^2 u - \operatorname{sn}^2 v + k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v}{1 - k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v}$

$= \dfrac{\operatorname{cn}^2 u + \operatorname{cn}^2 v}{1 - k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v} - 1 = 1 - \dfrac{\operatorname{sn}^2 u \cdot \operatorname{dn}^2 v + \operatorname{sn}^2 v \cdot \operatorname{dn}^2 u}{1 - k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v}.$

**727.** $\operatorname{dn}(u+v) \cdot \operatorname{dn}(u-v)$

$$= \frac{1 - k^2 \operatorname{sn}^2 u - k^2 \operatorname{sn}^2 v + k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v}{1 - k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v}$$

$$= \frac{\operatorname{dn}^2 u + \operatorname{dn}^2 v}{1 - k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v} - 1.$$

**728.** $\operatorname{sn}(u \pm v) \operatorname{cn}(u \mp v) = \dfrac{\operatorname{sn} u \cdot \operatorname{cn} u \cdot \operatorname{dn} v \pm \operatorname{sn} v \cdot \operatorname{cn} v \cdot \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v}.$

**729.** $\operatorname{sn}(u \pm v) \operatorname{dn}(u \mp v) = \dfrac{\operatorname{sn} u \cdot \operatorname{dn} u \cdot \operatorname{cn} v \pm \operatorname{sn} v \cdot \operatorname{dn} v \cdot \operatorname{cn} u}{1 - k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v}.$

**730.** $\operatorname{cn}(u \pm v) \operatorname{dn}(u \mp v) = \dfrac{\operatorname{cn} u \cdot \operatorname{dn} u \cdot \operatorname{cn} v \cdot \operatorname{dn} v \mp k'^2 \operatorname{sn} u \cdot \operatorname{sn} v}{1 - k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v}.$

**731.** $[1 \pm \operatorname{sn}(u+v)][1 \pm \operatorname{sn}(u-v)] = \dfrac{(\operatorname{cn} v \pm \operatorname{sn} u \cdot \operatorname{dn} v)^2}{1 - k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v}.$

**732.** $\operatorname{sn}(ui, k) = i \operatorname{sn}(u, k')/\operatorname{cn}(u, k').$

**733.** $\operatorname{cn}(ui, k) = 1/\operatorname{cn}(u, k').$

**734.** $\operatorname{dn}(ui, k) = \operatorname{dn}(u, k')/\operatorname{cn}(u, k').$

## D. — Bessel's Functions.

**735.** $J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots.$

**736.** $K_0(x) = J_0(x) \cdot \log x + \frac{x^2}{2^2} - \frac{x^4 \cdot \Omega_2}{2^2 \cdot 4^2} + \frac{x^6 \cdot \Omega_3}{2^2 \cdot 4^2 \cdot 6^2} - \cdots.$

$[\Omega_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + 1/k.]$

**737.** $J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{n+2k}}{2^{n+2k} \cdot k! \, \Gamma(n+k+1)}.$

[When $n$ is an integer, 819 may be used.]

**738.** $K_n(x) = J_n(x) \cdot \log x - \frac{x^{-n}}{2^{1-n}} \sum_{0}^{n-1} \frac{(n-k-1)! \, x^{2k}}{2^{2k} \cdot k!}$

$$- \frac{x^n}{2^{1+n}} \sum_{0}^{\infty} \frac{(-1)^k}{(n+k)! \, k!} \left[ \Omega_k + \Omega_{k+n} \left( \frac{x}{2} \right)^{2k} \right].$$

**739.** According as $n$ is or is not an integer, $A \cdot J_n(x) + B \cdot K_n(x)$, or $A \cdot J_n(x) + B \cdot J_{-n}(x)$ is a particular solution of Bessel's equation,

$$\frac{d^2z}{dx^2} + \frac{1}{x} \cdot \frac{dz}{dx} + \left( 1 - \frac{n^2}{x^2} \right) z = 0.$$

**740.** $dJ_0(x)/dx = -J_1(x)$; $d[x^n \cdot J_n(x)]/dx = x^n \cdot J_{n-1}(x)$, if $n > \frac{1}{2}$; $d[x^{-n} \cdot J_n(x)]/dx = -x^{-n} \cdot J_{n+1}(x)$, if $n > -\frac{1}{2}$.

**741.** $J_{n-1}(x) - J_{n+1}(x) = 2 \cdot dJ_n(x)/dx$;
$2n \cdot J_n(x) = x \cdot J_{n-1}(x) + x \cdot J_{n+1}(x)$.

When $x$ is large it is sometimes convenient to compute approximate numerical values of $J_n(x)$ by means of the semi-convergent series,

**742.** $J_n(x) = \sqrt{\frac{2}{\pi x}} \left[ P_n \cdot \cos \left\{ \frac{(2n+1)\pi}{4} - x \right\} + Q_n \cdot \sin \left\{ \frac{(2n+1)\pi}{4} - x \right\} \right].$

**743.** $P_n = 1 - \frac{(4n^2-1)(4n^2-9)}{2! \, (8x)^2} + \frac{(4n^2-1)(4n^2-9)(4n^2-25)(4n^2-49)}{4! \, (8x)^4} - \cdots$

**744.** $Q_n = \frac{4n^2-1}{8x} - \frac{(4n^2-1)(4n^2-9)(4n^2-25)}{3! \, (8x)^3} + \cdots.$

## E. — Series and Products.

[The expression in brackets attached to an infinite series shows values of the variable which lie within the interval of convergence. If a series is convergent for all finite values of $x$, the expression $[x^2 < \infty]$ is used.]

**745.** $(a+b)^n = a^n + na^{n-1}b$

$$+\frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots + \frac{n!\,a^{n-k}b^k}{(n-k)!\,k!} + \cdots. \quad [b^2 < a^2.]$$

**746.** $(a-bx)^{-1} = \frac{1}{a}\left[1 + \frac{bx}{a} + \frac{b^2x^2}{a^2} + \frac{b^3x^3}{a^3} + \cdots\right]. \quad [b^2x^2 < a^2.]$

**747.** $(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)}{2!}x^2$

$$\pm \frac{n(n-1)(n-2)x^3}{3!} + \cdots + \frac{(\pm 1)^k\, n!\, x^k}{(n-k)!\,k!} + \cdots.$$

$[x^2 < 1.]$

**748.** $(1 \pm x)^{-n} = 1 \mp nx + \frac{n(n+1)}{2!}x^2$

$$\mp \frac{n(n+1)(n+2)x^3}{3!} + \cdots (\mp)^k \frac{(n+k-1)!\,x^k}{(n-1)!\,k!} + \cdots.$$

$[x^2 < 1.]$

**749.** $(1 \pm x)^{\frac{1}{2}} = 1 \pm \tfrac{1}{2}x - \frac{1\cdot 1}{2\cdot 4}x^2 \pm \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}x^3$

$$- \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^4 \pm \cdots. \qquad [x^2 < 1.]$$

**750.** $(1 \pm x)^{-\frac{1}{2}} = 1 \mp \tfrac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 \mp \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3$

$$+ \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^4 \mp \cdots. \qquad [x^2 < 1.]$$

**751.** $(1 \pm x)^{\frac{1}{3}} = 1 \pm \tfrac{1}{3}x - \frac{1\cdot 2}{3\cdot 6}x^2 \pm \frac{1\cdot 2\cdot 5}{3\cdot 6\cdot 9}x^3$

$$- \frac{1\cdot 2\cdot 5\cdot 8}{3\cdot 6\cdot 9\cdot 12}x^4 \pm \cdots. \qquad [x^2 < 1.]$$

752. $(1 \pm x)^{-\frac{1}{3}} = 1 \mp \frac{1}{3}x + \frac{1 \cdot 4}{3 \cdot 6}x^2 \mp \frac{1 \cdot 4 \cdot 7}{3 \cdot 6 \cdot 9}x^3 + \frac{1 \cdot 4 \cdot 7 \cdot 10}{3 \cdot 6 \cdot 9 \cdot 12}x^4 \mp \cdots$ $[x^2 < 1.]$

753. $(1 \pm x^2)^{\frac{1}{2}} = 1 \pm \frac{1}{2}x^2 - \frac{x^4}{2 \cdot 4} \pm \frac{1 \cdot 3 x^6}{2 \cdot 4 \cdot 6} - \frac{1 \cdot 3 \cdot 5 x^8}{2 \cdot 4 \cdot 6 \cdot 8} \pm \cdots$ $[x^2 < 1.]$

754. $(1 \pm x^2)^{-\frac{1}{2}} = 1 \mp \frac{1}{2}x^2 + \frac{1 \cdot 3}{2 \cdot 4}x^4 \mp \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^6 + \cdots$ $[x^2 < 1.]$

755. $(1 \pm x)^{-1} = 1 \mp x + x^2 \mp x^3 + x^4 \mp x^5 + \cdots$ $[x^2 < 1.]$

756. $(1 \pm x)^{\frac{3}{2}} = 1 \pm \frac{3}{2}x + \frac{3 \cdot 1}{2 \cdot 4}x^2 \mp \frac{3 \cdot 1 \cdot 1}{2 \cdot 4 \cdot 6}x^3 + \frac{3 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6 \cdot 8}x^4 \mp \frac{3 \cdot 1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10}x^5 + \cdots$ $[x^2 < 1.]$

757. $(1 \pm x)^{-\frac{3}{2}} = 1 \mp \frac{3}{2}x + \frac{3 \cdot 5}{2 \cdot 4}x^2 \mp \frac{3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6}x^3 + \cdots$ $[x^2 < 1.]$

758. $(1 \pm x)^{-2} = 1 \mp 2x + 3x^2 \mp 4x^3 + 5x^4 \mp 6x^5 + \cdots$ $[x^2 < 1.]$

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759. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ $[x^2 < \infty.]$

760. $a^x = 1 + x \log a + \frac{(x \log a)^2}{2!} + \frac{(x \log a)^3}{3!} + \cdots$ $[x^2 < \infty.]$

761. $\frac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$ $[x^2 < \infty.]$

762. $\frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$ $[x^2 < \infty.]$

763. $e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots$ $[x^2 < \infty.]$

A series of numbers, $B_1$, $B_2$, $B_3 \cdots$, of odd and even orders, which appear in the developments of many functions, may be computed by means of the equations,

$$B_{2n} - \frac{2n(2n-1)}{2!} B_{2n-2}$$

$$+ \frac{2n(2n-1)(2n-2)(2n-3)}{4!} B_{2n-4} - \cdots + (-1)^n = 0.$$

$$\frac{2^{2n}(2^{2n}-1)}{2n} B_{2n-1} = (2n-1) B_{2n-2}$$

$$- \frac{(2n-1)(2n-2)(2n-3)}{3!} B_{2n-4} + \cdots + (-1)^{n-1}.$$

Whence $B_1 = \frac{1}{6}$, $B_2 = 1$, $B_3 = \frac{1}{30}$, $B_4 = 5$, $B_5 = \frac{1}{42}$, $B_6 = 61$, $B_7 = \frac{1}{30}$, $B_8 = 1385$, $B_9 = \frac{5}{66}$, $B_{10} = 50521$, $B_{11} = \frac{691}{2730}$, $B_{12} = 2702765$, $B_{13} = \frac{7}{6}$, etc. The $B$'s of odd orders are called Bernoulli's Numbers; those of even orders, Euler's Numbers. What are here denoted by $B_{2n-1}$ and $B_{2n}$ are sometimes represented by $B_n$ and $E_n$, respectively,

$$\frac{B_{2n-1}}{(2n)!} = \frac{2}{(2^{2n}-1)\pi^{2n}} \left[ 1 + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \cdots \right],$$

$$\frac{B_{2n}}{(2n)!} = \frac{2^{2n+2}}{\pi^{2n+1}} \left[ 1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \frac{1}{7^{2n+1}} + \cdots \right].$$

**764.** $$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{B_1 x^2}{2!} - \frac{B_3 x^4}{4!} + \frac{B_5 x^6}{6!} - \frac{B_7 x^8}{8!} + \cdots.$$

$[x < 2\pi.]$

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**765.** $$\log x = (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \cdots.$$

$[2 > x > 0.]$

**766.** $$\log x = \frac{x-1}{x} + \tfrac{1}{2}\left(\frac{x-1}{x}\right)^2 + \tfrac{1}{3}\left(\frac{x-1}{x}\right)^3 + \cdots.$$

$[x > \tfrac{1}{2}.]$

**767.** $\log x = 2\left[\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \cdots\right]$.

$[x > 0.]$

**768.** $\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$. $[x^2 < 1.]$

**769.** $\log\left(\frac{1+x}{1-x}\right) = 2[x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \cdots]$. $[x^2 < 1.]$

**770.** $\log\left(\frac{x+1}{x-1}\right) = 2\left[\frac{1}{x} + \frac{1}{3}\left(\frac{1}{x}\right)^3 + \frac{1}{5}\left(\frac{1}{x}\right)^5 + \cdots\right]$. $[x^2 > 1.]$

**771.** $\log(x + \sqrt{1+x^2}) = x - \frac{1x^3}{6} + \frac{1\cdot 3\,x^5}{2\cdot 4\cdot 5} - \frac{1\cdot 3\cdot 5\,x^7}{2\cdot 4\cdot 6\cdot 7} + \cdots$.

$[x^2 < 1.]$

Series for denary and other logarithms can be obtained from the foregoing developments by aid of the equations,

$$\log_a x = \log_e x \cdot \log_a e,\ \log_e x = \log_a x \cdot \log_e a,$$
$$\log_e(-z) = (2n+1)\pi i + \log_e z.$$

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**772.** $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$. $[x^2 < \infty.]$

**773.** $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = 1 - \operatorname{versin} x$. $[x^2 < \infty.]$

**774.** $\tan x = x + \frac{x^3}{3} + \frac{2\,x^5}{15} + \frac{17\,x^7}{315} + \frac{62\,x^9}{2835}$

$$+ \cdots + \frac{2^{2n}(2^{2n}-1)\,B_{2n-1}x^{2n-1}}{(2n)!} + \cdots. \quad [x^2 < \tfrac{1}{4}\pi^2.]$$

**775.** $\operatorname{ctn} x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2\,x^5}{945} - \frac{x^7}{4725}$

$$- \cdots - \frac{B_{2n-1}(2x)^{2n}}{x\,(2n)!} - \cdots. \quad [x^2 < \pi^2.]$$

**776.** $\sec x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \cdots + \frac{B_{2n}x^{2n}}{(2n)!} + \cdots . \left[x^2 < \frac{\pi^2}{4}.\right]$

**777.** $\csc x = \frac{1}{x} + \frac{x}{3!} + \frac{7x^3}{3\cdot 5!} + \frac{31x^5}{3\cdot 7!}$

$$+ \cdots + \frac{2(2^{2n+1}-1)}{(2n+2)!} B_{2n+1} x^{2n+1} + \cdots . \qquad [x^2 < \pi^2.]$$

**778.** $\sin^{-1} x = x + \frac{x^3}{6} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7}$

$$+ \cdots = \tfrac{1}{2}\pi - \cos^{-1} x. \qquad [x^2 < 1.]$$

**779.** $\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots = \tfrac{1}{2}\pi - \operatorname{ctn}^{-1} x.$

$[x^2 < 1.]$

**780.** $\tan^{-1} x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \cdots . \qquad [x^2 > 1.]$

**781.** $\sec^{-1} x = \frac{\pi}{2} - \frac{1}{x} - \frac{1}{6x^3} - \frac{1\cdot 3}{2\cdot 4\cdot 5x^5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7x^7} - \cdots$

$$= \tfrac{1}{2}\pi - \csc^{-1} x. \qquad [x^2 > 1.]$$

**782.** $\log \sin x = \log x - \frac{1}{6}x^2 - \frac{1}{180}x^4 - \frac{1}{2835}x^6$

$$- \cdots - \frac{2^{2n-1} B_{2n-1} x^{2n}}{n(2n)!} - \cdots . \qquad [x^2 < \pi^2.]$$

**783.** $\log \cos x = -\frac{1}{2}x^2 - \frac{1}{12}x^4 - \frac{1}{45}x^6 - \frac{1}{2520}x^8$

$$- \cdots - \frac{2^{2n-1}(2^{2n}-1) B_{2n-1} x^{2n}}{n(2n)!} - \cdots . \qquad [x^2 < \tfrac{1}{4}\pi^2.]$$

**784.** $\log \tan x = \log x + \frac{1}{3}x^2 + \frac{7}{90}x^4 + \frac{62}{2835}x^6$

$$+ \cdots + \frac{(2^{2n-1}-1)\,2^{2n} B_{2n-1} x^{2n}}{n(2n)!} + \cdots . \qquad [x^2 < \tfrac{1}{4}\pi^2.]$$

**785.** $e^{\sin x} = 1 + x + \frac{x^2}{2!} - \frac{3x^4}{4!} - \frac{8x^5}{5!} - \frac{3x^6}{6!} + \frac{56x^7}{7!} + \cdots .$

$[x^2 < \infty.]$

**786.** $e^{\cos x} = e\left(1 - \frac{x^2}{2!} + \frac{4\,x^4}{4!} - \frac{31\,x^6}{6!} + \cdots\right)$. $[x^2 < \infty.]$

**787.** $e^{\tan x} = 1 + x + \frac{x^2}{2!} + \frac{3\,x^3}{3!} + \frac{9\,x^4}{4!} + \frac{37\,x^5}{5!} + \cdots$. $[x^2 < \frac{1}{4}\pi^2.]$

**788.** $e^{\sin^{-1}x} = 1 + x + \frac{x^2}{2!} + \frac{2\,x^3}{3!} + \frac{5\,x^4}{4!} + \cdots$. $[x^2 < 1.]$

**789.** $e^{\tan^{-1}x} = 1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{7\,x^4}{24} - \cdots$. $[x^2 < 1.]$

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**790.** $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$. $[x^2 < \infty.]$

**791.** $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots$. $[x^2 < \infty.]$

**792.** $\tanh x = (2^2 - 1)\,2^2 B_1 \frac{x}{2!} - (2^4 - 1)\,2^4 B_3 \frac{x^3}{4!} + \cdots$

$= \Sigma[(-1)^{n-1}\,2^{2n}(2^{2n} - 1)\,B_{2n-1}\,x^{2n-1}/(2\,n)!]$. $[x^2 < \frac{1}{4}\pi^2.]$

**793.** $\operatorname{ctnh} x = \frac{1}{x}(1 + \Sigma[(-1)^{n-1}\,2^{2n}\,B_{2n-1}\,x^{2n}/(2\,n)!])$. $[x^2 < \pi^2.]$

**794.** $\operatorname{sech} x = 1 + \Sigma[(-1)^n\,B_{2n}\,x^{2n}/(2\,n)!]$. $[x^2 < \frac{1}{4}\pi^2.]$

**795.** $\operatorname{csch} x = \frac{1}{x} - (2 - 1)\,2\,B_1 \frac{x}{2!} + (2^3 - 1)\,2\,B_3 \frac{x^3}{4!} - \cdots$

$= \frac{1}{x}(1 + 2\,\Sigma[(-1)^n(2^{2n-1} - 1)\,B_{2n-1}\,x^{2n}/(2\,n)!])$. $[x^2 < \pi^2.]$

**796.** $\sinh^{-1} x = x - \frac{1}{6}\,x^3 + \frac{1 \cdot 3 \cdot x^5}{2 \cdot 4 \cdot 5} - \frac{1 \cdot 3 \cdot 5 \cdot x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots$. $[x^2 < 1.]$

797. $\tanh^{-1}x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots.$ $[x^2 < 1.]$

798. $\operatorname{ctnh}^{-1}x = \frac{1}{x} + \frac{1}{3\,x^3} + \frac{1}{5\,x^5} + \cdots.$ $[x^2 > 1.]$

799. $\operatorname{csch}^{-1}x = \frac{1}{x} - \frac{1}{2\cdot 3\cdot x^3} + \frac{1\cdot 3}{2\cdot 4\cdot 5\cdot x^5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7\cdot x^7} + \cdots.$ $[x^2 > 1.]$

800. $\int_0^x e^{-x^2}\,dx = x - \frac{1}{3}x^3 + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots.$ $[x^2 < \infty.]$

801. $\int_0^x \cos(x^2)\,dx = x - \frac{x^5}{5\cdot 2!} + \frac{x^9}{9\cdot 4!} - \frac{x^{13}}{13\cdot 6!} + \cdots.$ $[x^2 < \infty.]$

802. $\int_0^1 \frac{x^{a-1}dx}{1+x^b} = \frac{1}{a} - \frac{1}{a+b} + \frac{1}{a+2\,b} - \frac{1}{a+3\,b} + \cdots.$

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803. $f(x+h) = f(x) + h\cdot f'(x+\theta h).$

804. $f(x+h) = f(x) + h\cdot f'(x) + \frac{h^2}{2!}f''(x) + \cdots + \frac{h^n}{n!}\cdot f^n(x+\theta h).$

805. $f(x+h) = f(x) + h\cdot f'(x) + \frac{h^2}{2!}f''(x) + \cdots + \frac{h^n}{(n-1)!}\cdot(1-\theta)^{n-1}\cdot f^n(x+\theta h)$

806. $f(x+h, y+k) = f(x, y) + hf'_x(x+\theta h, y+\theta k) + kf'_y(x+\theta h, y+\theta k).$

807. $f(x+h, y+k) = f(x, y) + \left(h\frac{\partial f(x,y)}{\partial x} + k\frac{\partial f(x,y)}{\partial y}\right) + \frac{1}{2!}\left(h^2\frac{\partial^2 f(x,y)}{\partial x^2} + 2\,hk\frac{\partial^2 f(x,y)}{\partial x\cdot\partial y} + k^2\frac{\partial^2 f(x,y)}{\partial y^2}\right)$

$$+\frac{1}{3!}\left(h^3\frac{\partial^3 f(x,y)}{\partial x^3}+3h^2k\frac{\partial^3 f(x,y)}{\partial y\cdot\partial x^2}+3hk^2\frac{\partial^3 f(x,y)}{\partial x\cdot\partial y^2}\right.$$

$$\left.+k^3\frac{\partial f(x,y)}{\partial y^3}\right)+\cdots+R_n$$

$$=f(x,y)+(hD_x+kD_y)f(x,y)+\frac{1}{2!}(hD_x+kD_y)^2f(x,y)$$

$$+\cdots+\frac{1}{(n-1)!}(hD_x+kD_y)^{n-1}f(x,y)$$

$$+\frac{1}{n!}(hD_x+kD_y)^n f(x+\theta h,\ y+\theta k).$$

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**808.** $1=\frac{4}{\pi}\left[\sin\frac{\pi x}{c}+\frac{1}{3}\sin\frac{3\pi x}{c}+\frac{1}{5}\sin\frac{5\pi x}{c}+\cdots\right].$ $[0<x<c.]$

**809.** $x=\frac{2c}{\pi}\left[\sin\frac{\pi x}{c}-\frac{1}{2}\sin\frac{2\pi x}{c}+\frac{1}{3}\sin\frac{3\pi x}{c}-\cdots\right].$ $[-c<x<c.]$

**810.** $x=\frac{c}{2}-\frac{4c}{\pi^2}\left[\cos\frac{\pi x}{c}+\frac{1}{3^2}\cos\frac{3\pi x}{c}+\frac{1}{5^2}\cos\frac{5\pi x}{c}+\cdots\right].$ $[0<x<c.]$

**811.** $x^2=\frac{2c^2}{\pi^3}\left[\left(\frac{\pi^2}{1}-\frac{4}{1}\right)\sin\frac{\pi x}{c}-\frac{\pi^2}{2}\sin\frac{2\pi x}{c}\right.$

$$+\left(\frac{\pi^2}{3}-\frac{4}{3^3}\right)\sin\frac{3\pi x}{c}-\frac{\pi^2}{4}\sin\frac{4\pi x}{c}$$

$$\left.+\left(\frac{\pi^2}{5}-\frac{4}{5^3}\right)\sin\frac{5\pi x}{c}+\cdots\right].\qquad [0<x<c.]$$

**812.** $x^2=\frac{c^2}{3}-\frac{4c^2}{\pi^2}\left[\cos\frac{\pi x}{c}-\frac{1}{2^2}\cos\frac{2\pi x}{c}+\frac{1}{3^2}\cos\frac{3\pi x}{c}\right.$

$$\left.-\frac{1}{4^2}\cos\frac{4\pi x}{c}+\cdots\right].\qquad [-c<x<c.]$$

**813.** $\log \sin \tfrac{1}{2}x = -\log 2 - \cos x - \tfrac{1}{2}\cos 2x - \tfrac{1}{3}\cos 3x - \cdots.$

$[0 < x < \tfrac{1}{2}\pi.]$

**814.** $\log \cos \tfrac{1}{2}x = -\log 2 + \cos x - \tfrac{1}{2}\cos 2x + \tfrac{1}{3}\cos 3x - \cdots.$

$[0 < x < \tfrac{1}{2}\pi.]$

**815.** $f(x) = \tfrac{1}{2}b_0 + b_1 \cos\frac{\pi x}{c} + b_2 \cos\frac{2\pi x}{c} + \cdots$

$$+ a_1 \sin\frac{\pi x}{c} + a_2 \sin\frac{2\pi x}{c} + \cdots, \quad [-c < x < c.]$$

where $b_m = \frac{1}{c}\int_{-c}^{+c} f(a)\cos\frac{m\pi a}{c}\,da,$

$$a_m = \frac{1}{c}\int_{-c}^{+c} f(a)\sin\frac{m\pi a}{c}\,da.$$

---

**816.** $\sin\theta = \theta\left[1-\left(\frac{\theta}{\pi}\right)^2\right]\left[1-\left(\frac{\theta}{2\pi}\right)^2\right]\left[1-\left(\frac{\theta}{3\pi}\right)^2\right]\cdots.$

$[\theta^2 < \infty.]$

**817.** $\cos\theta = \left[1-\left(\frac{2\theta}{\pi}\right)^2\right]\left[1-\left(\frac{2\theta}{3\pi}\right)^2\right]\left[1-\left(\frac{2\theta}{5\pi}\right)^2\right]\cdots$

$[\theta^2 < \infty.]$

**818.** $\dfrac{2^2\cdot 4^2\cdot 6^2\cdots(2m)^2(2m+2)}{1^2\cdot 3^2\cdot 5^2\cdots(2m+1)^2} > \frac{\pi}{2}$

$$> \frac{2^2\cdot 4^2\cdot 6^2\cdots(2m)^2(2m+1)}{1^2\cdot 3^2\cdot 5^2\cdots(2m+1)^2}.$$

**819.** $J_n(x) = \dfrac{x^n}{2^n n!}\left\{1 - \dfrac{x^2}{2(2n+2)} + \dfrac{x^4}{2\cdot 4\,(2n+2)(2n+4)}\right.$

$$\left. - \frac{x^6}{2\cdot 4\cdot 6\,(2n+2)(2n+4)(2n+6)} + \cdots\right\}.$$

## F. — Derivatives.

820. $\frac{d(au)}{dx} = a\frac{du}{dx}.$

821. $\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$

822. $\frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}.$

823. $\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$

824. $\frac{df(u)}{dx} = \frac{df(u)}{du} \cdot \frac{du}{dx}.$

825. $\frac{d^2f(u)}{dx^2} = \frac{df}{du} \cdot \frac{d^2u}{dx^2} + \frac{d^2f}{du^2} \cdot \frac{du^2}{dx^2}.$

826. $\frac{dx^n}{dx} = nx^{n-1}.$

827. $\frac{de^x}{dx} = e^x.$

828. $\frac{da^u}{dx} = a^u \cdot \frac{du}{dx} \cdot \log_e a.$

829. $\frac{dx^x}{dx} = x^x(1 + \log_e x).$

830. $\frac{d(\log_a x)}{dx} = \frac{1}{x \cdot \log_e a} = \frac{\log_a e}{x}.$

831. $\frac{d \sin x}{dx} = \cos x.$

832. $\frac{d \cos x}{dx} = -\sin x.$

833. $\dfrac{d \tan x}{dx} = \sec^2 x.$

834. $\dfrac{d \operatorname{ctn} x}{dx} = -\csc^2 x.$

835. $\dfrac{d \sec x}{dx} = \tan x \cdot \sec x.$

836. $\dfrac{d \csc x}{dx} = -\operatorname{ctn} x \cdot \csc x.$

837. $\dfrac{d \sin^{-1} x}{dx} = \dfrac{1}{\sqrt{1-x^2}}.$

838. $\dfrac{d \cos^{-1} x}{dx} = \dfrac{-1}{\sqrt{1-x^2}}.$

839. $\dfrac{d \tan^{-1} x}{dx} = \dfrac{1}{1+x^2}.$

840. $\dfrac{d \operatorname{ctn}^{-1} x}{dx} = -\dfrac{1}{1+x^2}.$

841. $\dfrac{d \sec^{-1} x}{dx} = \dfrac{1}{x\sqrt{x^2-1}}.$

842. $\dfrac{d \csc^{-1} x}{dx} = -\dfrac{1}{x\sqrt{x^2-1}}.$

843. $\dfrac{d \sinh x}{dx} = \cosh x.$

844. $\dfrac{d \cosh x}{dx} = \sinh x.$

845. $\dfrac{d \tanh x}{dx} = \operatorname{sech}^2 x.$

846. $\dfrac{d \operatorname{ctnh} x}{dx} = -\operatorname{csch}^2 x.$

**847.** $\dfrac{d\ \operatorname{sech} x}{dx} = -\operatorname{sech} x \cdot \tanh x.$

**848.** $\dfrac{d\ \operatorname{csch} x}{dx} = -\operatorname{csch} x \cdot \operatorname{ctnh} x.$

**849.** $\dfrac{d\ \sinh^{-1} x}{dx} = \dfrac{1}{\sqrt{x^2+1}}.$

**850.** $\dfrac{d\ \cosh^{-1} x}{dx} = \dfrac{1}{\sqrt{x^2-1}}.$

**851.** $\dfrac{d\ \tanh^{-1} x}{dx} = \dfrac{1}{1-x^2}.$

**852.** $\dfrac{d\ \operatorname{ctnh}^{-1} x}{dx} = \dfrac{1}{1-x^2}.$

**853.** $\dfrac{d\ \operatorname{sech}^{-1} x}{dx} = \dfrac{-1}{x\sqrt{1-x^2}}.$

**854.** $\dfrac{d\ \operatorname{csch}^{-1} x}{dx} = \dfrac{-1}{x\sqrt{x^2+1}}.$

**855.** $\dfrac{d}{db}\int_a^b f(x)\,dx = f(b).$

**856.** $\dfrac{d}{da}\int_a^b f(x)\,dx = -f(a).$

**857.** $\dfrac{d}{dc}\int_a^b f(x,c)\,dx = \int_a^b D_c f(x,c)\cdot dx + f(b,c)\dfrac{db}{dc} - f(a,c)\dfrac{da}{dc}.$

**858.** $\dfrac{d^n(u\cdot v)}{dx^n} = v\cdot\dfrac{d^n u}{dx^n} + n\cdot\dfrac{dv}{dx}\cdot\dfrac{d^{n-1}u}{dx^{n-1}}$

$$+\frac{n(n-1)}{2!}\cdot\frac{d^2v}{dx^2}\cdot\frac{d^{n-2}u}{dx^{n-2}} + \cdots + u\frac{d^n v}{dx^n}.$$

**859.** If $f(x, y, z, \cdots)$ is a homogeneous function of the $n$th order, so that $f(\lambda x, \lambda y, \lambda z, \cdots) \equiv \lambda^n f(x, y, z, \cdots)$,

$$x\cdot D_x f + y\cdot D_y f + z\cdot D_z f + \cdots \equiv nf.$$

**860.** If $x = \phi(y)$,

$$\frac{dy}{dx} = \frac{1}{\phi'(y)}, \quad \frac{d^2y}{dx^2} = -\frac{\phi''(y)}{[\phi'(y)]^3},$$

$$\frac{d^3y}{dx^3} = \frac{3[\phi''(y)]^2 - \phi'(y)\cdot\phi'''(y)}{[\phi'(y)]^5}.$$

**861.** If $x = f(t)$ and $y = \phi(t)$,

$$\frac{dy}{dx} = \frac{\phi'(t)}{f'(t)}, \quad \frac{d^2y}{dx^2} = \frac{f'(t)\cdot\phi''(t) - f''(t)\cdot\phi'(t)}{[f'(t)]^3}.$$

**862.** If $f(x, y) = 0$,

$$\frac{dy}{dx} = -\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y} = -\frac{D_x f}{D_y f},$$

$$\frac{d^2y}{dx^2} = -\frac{D_x^2 f\cdot(D_y f)^2 - 2\,D_x D_y f\cdot D_x f\cdot D_y f + D_y^2 f\cdot(D_x f)^2}{(D_y f)^3}$$

**863.** If $y = f(u, v)$, $u = \phi(x)$, and $v = \psi(x)$,

$$\frac{df}{dx} = \frac{\partial f}{\partial u}\cdot\frac{du}{dx} + \frac{\partial f}{\partial v}\cdot\frac{dv}{dx} = u'\cdot D_u f + v'\cdot D_v f,$$

$$\frac{d^2f}{dx^2} = \frac{\partial^2 f}{\partial u^2}\cdot\left(\frac{du}{dx}\right)^2 + 2\,\frac{\partial^2 f}{\partial u\cdot\partial v}\cdot\frac{du}{dx}\cdot\frac{dv}{dx} + \frac{\partial^2 f}{\partial^2 v}\cdot\left(\frac{dv}{dx}\right)^2$$

$$+ \frac{\partial f}{\partial u}\cdot\frac{d^2u}{dx^2} + \frac{\partial f}{\partial v}\cdot\frac{d^2v}{dx^2}$$

$$= u'^2\cdot D_u^2 f + 2\,u'\cdot v'\cdot D_u D_v f + v'^2\cdot D_v^2 f$$

$$+ u''\cdot D_u f + v''\cdot D_v f.$$

**864.** If $f(x, y, z) = 0$, $D_x z = -D_x f / D_z f$,

$$D_x^2 z = -[D_x^2 f\cdot(D_z f)^2$$

$$- 2\,D_z f\cdot D_x f\cdot D_x D_y f + D_z^2 f(D_x f)^2]/(D_z f)^3,$$

$$D_x D_y z = -[D_x D_y f\cdot(D_z f)^2 - D_z f\,D_x f\cdot D_y D_z f$$

$$+ D_z f\cdot D_y f\cdot D_x D_z f + D_x f\cdot D_y f\cdot D_z^2 f]/(D_z f)^3.$$

**865.** If $V = \phi(u, v)$, $u = f_1(x, y)$, and $v = f_2(x, y)$,

$$D_x V = D_u\phi \cdot D_x u + D_v\phi \cdot D_x v,$$

$$D_x^2 V = D_u^2\phi \cdot (D_x u)^2 + D_v^2\phi \cdot (D_x v)^2 + 2\, D_u D_v \phi \cdot D_x u \cdot D_x v + D_u\phi\, D_x^2 u + D_v\phi \cdot D_x^2 v,$$

$$D_y D_x V = D_u^2\phi \cdot D_x u \cdot D_y u + D_v^2\phi \cdot D_x v \cdot D_y v + D_u D_v \phi (D_x v \cdot D_y u + D_x u \cdot D_y v) + D_u\phi \cdot D_x D_y u + D_v\phi \cdot D_x D_y v,$$

$$D_x^2 V + D_y^2 V = D_u^2\phi \cdot [(D_x u)^2 + (D_y u)^2] + D_v^2\phi \cdot [(D_x v)^2 + (D_y v)^2] + 2\, D_u D_v \phi \cdot [D_x u \cdot D_x v + D_y u \cdot D_y v] + D_u\phi \cdot [D_x^2 u + D_y^2 u] + D_v\phi \cdot [D_x^2 v + D_y^2 v].$$

In the special case, $u \equiv r \equiv \sqrt{x^2 + y^2}$, $v \equiv \theta \equiv \tan^{-1}(y/x)$, we have $D_r x = \cos\theta = x/\sqrt{x^2+y^2}$; $D_r y = \sin\theta = y/\sqrt{x^2+y^2}$;

$$D_\theta x = -r\sin\theta = -y; \quad D_\theta y = r\cos\theta = x;$$

$$D_x r = x/\sqrt{x^2+y^2} = \cos\theta; \quad D_y r = y/\sqrt{x^2+y^2} = \sin\theta;$$

$$D_x\theta = -y/(x^2+y^2) = -\sin\theta/r;$$

$$D_y\theta = x/(x^2+y^2) = \cos\theta/r; \text{ and}$$

$$D_x^2 V + D_y^2 V = D_r^2 V + \frac{1}{r}\cdot D_r V + \frac{1}{r^2}\cdot D_\theta^2 V.$$

**866.** If $V = \phi(u, v)$, $u = f_1(r, \theta)$, and $v = f_2(r, \theta)$,

$$D_r^2 V + \frac{1}{r}\cdot D_r V + \frac{1}{r^2}\cdot D_\theta^2 V = D_u^2 V \cdot \left[(D_r u)^2 + \frac{(D_\theta u)^2}{r^2}\right] + D_v^2 V \cdot \left[(D_r v)^2 + \frac{(D_\theta v)^2}{r^2}\right] + 2\, D_u D_v V \left[D_r u \cdot D_r v + \frac{D_\theta u \cdot D_\theta v}{r^2}\right] +$$

$$+ D_u V\left[D_r{}^2 u + \frac{1}{r}\cdot D_r u + \frac{1}{r^2}\cdot D_\theta{}^2 u\right]$$
$$+ D_v V\left[D_r{}^2 v + \frac{1}{r}\cdot D_r v + \frac{1}{r^2}\cdot D_\theta{}^2 v\right].$$

**867.** If $V = \phi(u, v, w)$, $u = f_1(x, y, z)$, $v = f_2(x, y, z)$, and $w = f_3(x, y, z)$,

$$D_x V = D_u V\cdot D_x u + D_v V\cdot D_x v + D_w V\cdot D_x w,$$
$$D_x{}^2 V = D_u{}^2 V\cdot (D_x u)^2 + D_v{}^2 V\cdot (D_x v)^2 + D_w{}^2 V\cdot (D_x w)^2$$
$$+ D_u V\cdot D_x{}^2 u + D_v V\cdot D_x{}^2 v + D_w V\cdot D_x{}^2 w$$
$$+ 2\,(D_u D_v V\cdot D_x u\cdot D_x v + D_u D_w V\cdot D_x u\cdot D_x w$$
$$+ D_v D_w V\cdot D_x v\cdot D_x w).$$
$$D_x{}^2 V + D_y{}^2 V + D_z{}^2 V = D_u{}^2 V\cdot[(D_x u)^2 + (D_y u)^2 + (D_z u)^2]$$
$$+ D_v{}^2 V\cdot[(D_x v)^2 + (D_y v)^2 + (D_z v)^2]$$
$$+ D_w{}^2 V[(D_x w)^2 + (D_y w)^2 + (D_z w)^2]$$
$$+ 2\,D_u D_v V\cdot[D_x u\cdot D_x v + D_y u\cdot D_y v + D_z u\cdot D_z v]$$
$$+ 2\,D_v D_w V\cdot[D_x v\cdot D_x w + D_y v\cdot D_y w + D_z v\cdot D_z w]$$
$$+ 2\,D_w D_u V\cdot[D_x w\cdot D_x u + D_y w\cdot D_y u + D_z w\cdot D_z u]$$
$$+ D_u V\cdot[D_x{}^2 u + D_y{}^2 u + D_z{}^2 u]$$
$$+ D_v V\cdot[D_x{}^2 v + D_y{}^2 v + D_z{}^2 v]$$
$$+ D_w V\cdot[D_x{}^2 w + D_y{}^2 w + D_z{}^2 w].$$

In particular, if

$$x \equiv r\sin\theta\cos\phi,\quad y \equiv r\sin\theta\sin\phi,\quad z \equiv r\cos\theta,$$

so that $u \equiv r^2 \equiv x^2 + y^2 + z^2$, $v \equiv \theta \equiv \tan^{-1}(\sqrt{x^2+y^2}/z)$,

$w \equiv \phi \equiv \tan^{-1}(y/x)$, we have

$$D_r z = \cos\theta = z/\sqrt{x^2+y^2+z^2};$$
$$D_r x = \sin\theta\cos\phi = x/\sqrt{x^2+y^2+z^2};$$

$$D_r y = \sin\theta \sin\phi = y/\sqrt{x^2+y^2+z^2};$$

$$D_\theta z = -r\sin\theta = -\sqrt{x^2+y^2};$$

$$D_\theta x = r\cos\theta\cos\phi = zx/\sqrt{x^2+y^2};$$

$$D_\theta y = r\cos\theta\sin\phi = zy/\sqrt{x^2+y^2};$$

$$D_\phi z = 0;$$

$$D_\phi x = -r\sin\theta\sin\phi = -y;$$

$$D_\phi y = r\sin\theta\cos\phi = x;$$

$$D_z r = z/r = \cos\theta;$$

$$D_z\theta = -\sqrt{x^2+y^2}/r^2 = -\sin\theta/r;$$

$$D_z\phi = 0;$$

$$D_x r = x/r = \sin\theta\cos\phi;$$

$$D_x\theta = xz/r^2\sqrt{x^2+y^2} = \cos\theta\cos\phi/r;$$

$$D_x\phi = -y/(x^2+y^2) = -\sin\phi/r\sin\theta;$$

$$D_y r = y/r = \sin\theta\sin\phi;$$

$$D_y\theta = zy/r^2\sqrt{x^2+y^2} = \cos\theta\sin\phi/r;$$

$$D_y\phi = x/(x^2+y^2) = \cos\phi/r\sin\theta;$$

$$(D_x r)^2 + (D_y r)^2 + (D_z r)^2 = 1;$$

$$(D_x\theta)^2 + (D_y\theta)^2 + (D_z\theta)^2 = 1/r^2;$$

$$(D_x\phi)^2 + (D_y\phi)^2 + (D_z\phi)^2 = 1/r^2\sin^2\theta;$$

$$(D_x V)^2 + (D_y V)^2 + (D_z V)^2 = (D_r V)^2 + \left(\frac{D_\theta V}{r}\right)^2 + \left(\frac{D_\phi V}{r\sin\theta}\right)^2;$$

$$D_x^2 V + D_y^2 V + D_z^2 V = \frac{1}{r^2\sin\theta}\left[D_r(r^2\cdot D_r V)\cdot\sin\theta + \frac{D_\phi^2 V}{\sin\theta} + D_\theta(\sin\theta\cdot D_\theta V)\right]$$

868. If $x = f_1(u, v)$, $y = f_2(u, v)$, $z = f_3(u, v)$,

$$D_x z = \frac{D_u f_3 \cdot D_v f_2 - D_v f_3 \cdot D_u f_2}{D_u f_1 \cdot D_v f_2 - D_v f_1 \cdot D_u f_2},$$

$$D_y z = \frac{D_v f_3 \cdot D_u f_1 - D_u f_3 \cdot D_v f_1}{D_u f_1 \cdot D_v f_2 - D_v f_1 \cdot D_u f_2}.$$

869. If $x = f(z, u)$, and $y = \phi(z, u)$,

$$D_x z = D_u \phi / (D_z f \cdot D_u \phi - D_z \phi \cdot D_u f),$$

$$D_y z = D_u f / (D_z \phi \cdot D_u f - D_z f \cdot D_u \phi).$$

870. If $F_1(x, y, z, u, v) = 0$,

$F_2(x, y, z, u, v) = 0$, and $F_3(x, y, z, u, v) = 0$,

$$D_x z \cdot \begin{vmatrix} D_z F_1 & D_u F_1 & D_v F_1 \\ D_z F_2 & D_u F_2 & D_v F_2 \\ D_z F_3 & D_u F_3 & D_v F_3 \end{vmatrix} = - \begin{vmatrix} D_x F_1 & D_u F_1 & D_v F_1 \\ D_x F_2 & D_u F_2 & D_v F_2 \\ D_x F_3 & D_u F_3 & D_v F_3 \end{vmatrix}.$$

871. If $F_1(x, y, z) = 0$, and $F_2(x, y, z) = 0$,

$$\frac{dy}{D_z F_1 \cdot D_x F_2 - D_z F_2 \cdot D_x F_1} = \frac{dz}{D_x F_1 \cdot D_y F_2 - D_x F_2 \cdot D_y F_1} = \frac{dx}{D_y F_1 \cdot D_z F_2 - D_y F_2 \cdot D_z F_1}.$$

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If each of the quantities $y_1, y_2, y_3, \cdots y_n$ is a function of the $n$ variables $x_1, x_2, x_3, \cdots x_n$, the determinant,

$$\begin{vmatrix} D_{x_1} y_1 & D_{x_2} y_1 & D_{x_3} y_1 \cdots \\ D_{x_1} y_2 & D_{x_2} y_2 & D_{x_3} y_2 \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ D_{x_1} y_n & D_{x_2} y_n & D_{x_3} y_n \cdots D_{x_n} y_n \end{vmatrix}$$

is called the *functional determinant* or the *Jacobian* of the $y$'s with respect to the $x$'s and is denoted by the expression,

$$\frac{\partial(y_1, y_2, y_3, \cdots y_n)}{\partial(x_1, x_2, x_3, \cdots x_n)}, \text{ or by } J(y_1, y_2, \cdots y_n).$$

**872.** $$\frac{\partial(y_1, y_2, y_3, \cdots y_n)}{\partial(x_1, x_2, x_3, \cdots x_n)} \cdot \frac{\partial(x_1, x_2, x_3, \cdots x_n)}{\partial(y_1, y_2, y_3, \cdots y_n)} \equiv 1.$$

**873.** $$\frac{\partial(y_1, y_2, y_3, \cdots y_n)}{\partial(z_1, z_2, z_3, \cdots z_n)} \cdot \frac{\partial(z_1, z_2, z_3, \cdots z_n)}{\partial(x_1, x_2, x_3, \cdots x_n)} \equiv \frac{\partial(y_1, y_2, y_3, \cdots y_n)}{\partial(x_1, x_2, x_3, \cdots x_n)}.$$

If the $y$'s are not all independent but are connected by an equation of the form $\phi(y_1, y_2, y_3, \cdots y_n) = 0$, the Jacobian of the $y$'s with respect to the $x$'s vanishes identically; and, conversely, if the Jacobian vanishes identically, the $y$'s are connected by one or more relations of the above-mentioned form.

---

The *directional derivative* of any scalar point function, $u$, at any point, $P$, in any fixed direction $PQ'$, is the limit, as $PQ$ approaches zero, of the ratio of $u_Q - u_P$ to $PQ$, where $Q$ is a point on the straight line $PQ'$ between $P$ and $Q'$. The *gradient*, $h_u$, of the function $u$ at $P$ is the directional derivative of $u$ at $P$ taken in the direction in which $u$ increases most rapidly. This direction is normal to the surface of constant $u$ which passes through $P$.

**874.** $h_u^2 \equiv (D_x u)^2 + (D_y u)^2 + (D_z u)^2$.

The directional derivative of any scalar point function at any point in any given direction is evidently equal to the product of the gradient and the cosine of the angle between the given direction and that in which the function increases most rapidly.

The *normal derivative*, at any point, $P$, of a point function $u$, taken with respect to another point function $v$, is the limit as $PQ$ approaches zero of the ratio of $u_Q - u_P$ to $v_Q - v_P$, where $Q$ is a point so chosen on the normal at $P$ of the surface of constant $v$ which passes through $P$, that $v_Q - v_P$ is positive. If $(u, v)$ denotes the angle between the directions in which $u$ and $v$ increase most rapidly, the normal derivatives of $u$ with respect to $v$, and of $v$ with respect to $u$ may be written

$$h_u \cos(u, v) \div h_v, \text{ and } h_v \cdot \cos(u, v) \div h_u$$

respectively. If $h_u = h_v$, these derivatives are equal.

## G. — Miscellaneous Formulas.

If $s$ is a plane analytic closed curve, $n$ its normal drawn from within outwards, and $dA$ the element of plane area within $s$, the usual integral transformation formulas for the functions $u$ and $v$ which, with their derivatives of the first order, are continuous everywhere within $s$, may be written —

**875.** $$\int u \cdot \cos(x, n)\, ds = \iint D_x u \cdot dA.$$

**876.** $$\int [u \cdot \cos(x, n) + v \cdot \cos(y, n)]\, ds = \iint (D_x u + D_y v)\, dA.$$

**877.** $$\int D_n u \cdot ds = \iint (D_x{}^2 u + D_y{}^2 u)\, dA.$$

**878.** $$\iint (D_x u \cdot D_x v + D_y u \cdot D_y v)\, dA$$
$$= \int u \cdot D_n v \cdot ds - \iint u (D_x{}^2 v + D_y{}^2 v)\, dA$$
$$= \int v \cdot D_n u \cdot ds - \iint v (D_x{}^2 u + D_y{}^2 v)\, dA.$$

**879.** $$\iint \lambda (D_x u \cdot D_x v + D_y u \cdot D_y v)\, dA = \int \lambda \cdot u \cdot D_n v \cdot ds$$
$$- \iint u [D_x (\lambda \cdot D_x v) + D_y (\lambda \cdot D_y v)]\, dA.$$

If $\xi$ and $\eta$ are two analytic functions which define a set of orthogonal curvilinear coördinates, and if $(\xi, n)$ and $(\eta, n)$ represent the angles between $n$ and the directions in which $\xi$ and $\eta$, respectively, increase most rapidly.

**880.** $$\int\int h_\xi \cdot h_\eta \cdot D_\eta\left(\frac{u}{h_\xi}\right) dA = \int u \cdot \cos(\eta, n)\, ds.$$

**881.** $$\int\int h_\xi \cdot h_\eta \cdot D_\xi\left(\frac{u}{h_\eta}\right) dA = \int u \cdot \cos(\xi, n)\, ds.$$

**882.** If $r$ is the distance from a fixed point, $Q$, in the coördinate plane,

$$\int \frac{\cos(r, n)\, ds}{r} = 0,\ \pi,\ \text{or}\ 2\pi,\ \text{according as}\ Q\ \text{is without, on, or within}\ s.$$

---

If $S$ is an analytic closed surface, $n$ its normal drawn from within outwards, and $d\tau$ the element of volume shut in by $S$, the usual integral transformation formulas may be written —

**883.** $$\int\int u \cos(x, n)\, dS = \int\int\int D_x u \cdot d\tau.$$

**884.** $$\int\int [u \cos(x, n) + v \cos(y, n) + w \cos(z, n)]\, dS = \int\int\int (D_x u + D_y v + D_z w)\, d\tau.$$

**885.** $$\int\int D_n u \cdot ds = \int\int\int (D_x^2 u + D_y^2 u + D_z^2 u)\, d\tau.$$

**886.** $$\int\int\int (D_x u \cdot D_x v + D_y u \cdot D_y v + D_z u \cdot D_z v)\, d\tau$$
$$= \int\int u \cdot D_n v \cdot dS - \int\int\int u (D_x^2 v + D_y^2 v + D_z^2 v)\, d\tau$$
$$= \int\int v \cdot D_n \boldsymbol{u} \cdot \boldsymbol{dS} - \int\int\int v (D_x^2 u + D_y^2 u + D_z^2 u)\, d\tau.$$

887. $$\iiint \lambda (D_x u \cdot D_x v + D_y u \cdot D_y v + D_z u \cdot D_z v)\, d\tau$$
$$= \iint \lambda \cdot v \cdot D_n u \cdot dS$$
$$- \iiint v \left[ D_x (\lambda D_x u) + D_y (\lambda D_y u) + D_z (\lambda D_z u) \right] d\tau.$$

If $\xi$, $\eta$, $\zeta$ are three analytic functions which define a system of orthogonal curvilinear coördinates,

888. $$\iiint h_\xi \cdot h_\eta \cdot h_\zeta \cdot D_\xi \left( \frac{u}{h_\eta \cdot h_\zeta} \right) d\tau = \iint u \cdot \cos(\xi, n)\, dS.$$

889. $$\iiint h_\xi \cdot h_\eta \cdot h_\zeta \cdot D_\eta \left( \frac{u}{h_\xi \cdot h_\zeta} \right) d\tau = \iint u \cdot \cos(\eta, n)\, dS.$$

890. $$\iiint h_\xi \cdot h_\eta \cdot h_\zeta \cdot D_\zeta \left( \frac{u}{h_\xi \cdot h_\eta} \right) d\tau = \iint u \cdot \cos(\zeta, n)\, dS.$$

891. If $r$ is the distance from a fixed point, $Q$,

$$\int \frac{\cos(r, n)}{r^2}\, dS = 0,\ 2\pi, \text{ or } 4\pi$$ according as $Q$ is without, on, or within $S$.

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Stokes's Theorem. — The line integral, taken around a closed curve, of the tangential component of a vector point function, is equal to the surface integral, taken over a surface bounded by the curve, of the normal component of the curl of the vector, the direction of integration around the curve forming a right-handed screw rotation about the normals.

If $X$, $Y$, $Z$ are the components of the vector,

892. $$\int (X\,dx + Y\,dy + Z\,dz) = \iint \left[ (D_y Z - D_z Y) \cos(x, n) \right.$$
$$+ (D_z X - D_x Z) \cos(y, n)$$
$$\left. + (D_x Y - D_y X) \cos(z, n) \right] dS.$$

Equations 893 to 897 give Poisson's Equation in orthogonal Cartesian, in cylindrical, in spherical, and in orthogonal curvilinear coördinates.

**893.** $\overline{\nabla}^2 V \equiv D_x{}^2 V + D_y{}^2 V + D_z{}^2 V = -4\pi\rho.$

**894.** $\frac{1}{r}\cdot D_r(r\cdot D_r V) + \frac{1}{r^2}\cdot D_\theta{}^2 V + D_z{}^2 V = -4\pi\rho.$

**895.** $\sin\theta\cdot D_r(r^2\cdot D_r V) + \frac{D_\phi{}^2 V}{\sin\theta} + D_\theta(\sin\theta\cdot D_\theta V) = -4\pi\rho r^2\sin\theta.$

**896.** $h_\xi{}^2\cdot D_\xi{}^2 V + h_\eta{}^2\cdot D_\eta{}^2 V + h_\zeta{}^2\cdot D_\zeta{}^2 V + D_\xi V\cdot\overline{\nabla}^2\xi + D_\eta V\cdot\overline{\nabla}^2\eta + D_\zeta V\cdot\overline{\nabla}^2\zeta = -4\pi\rho.$

**897.** $h_\xi\cdot h_\eta\cdot h_\zeta\left\{D_\xi\left(\frac{h_\xi}{h_\eta h_\zeta}\cdot D_\xi V\right) + D_\eta\left(\frac{h_\eta}{h_\xi h_\zeta}\cdot D_\eta V\right) + D_\zeta\left(\frac{h_\zeta}{h_\xi h_\eta}\cdot D_\zeta V\right)\right\} = -4\pi\rho$

## H. — Certain Constants.

$$\pi = 3.14159\ 26535\ 89793$$
$$\log_{10}\pi = 0.49714\ 98726\ 94134$$
$$\frac{1}{\pi} = 0.31830\ 98861\ 83791$$
$$\pi^2 = 9.86960\ 44010\ 89359$$
$$\sqrt{\pi} = 1.77245\ 38509\ 05516$$
$$\log_{10} 2 = 0.30102\ 99956\ 63981$$
$$e = 2.71828\ 18284\ 59045$$
$$\log_{10} e = 0.43429\ 44819\ 03252$$
$$\log_e 10 = 2.30258\ 50929\ 94046$$
$$\log_e 2 = 0.69314\ 71805\ 59945$$
$$\log_{10}\log_{10} e = 9.63778\ 43113\ 00537$$
$$\log_e \pi = 1.14472\ 98858\ 49400$$

## I. — General Formulas of Integration.

$F$ and $f$ represent functions of $x$, and $F'$, $f'$, $F''$, $f''$, their first and second derivatives with respect to $x$.

898. $\int F' \cdot f \cdot dx = F \cdot f - \int F \cdot f' \cdot dx.$

899. $\int (F)^n \cdot F' \cdot dx = (F)^{n+1}/(n+1).$

900. $\int (aF+b)^n \cdot F' \cdot dx = (aF+b)^{n+1}/a\,(n+1).$

901. $\int (F+f)^n \cdot dx = \int F(F+f)^{n-1} dx + \int f(F+f)^{n-1} dx.$

902. $\int F'/(F)^n \cdot dx = -1/(n-1)(F)^{n-1}, \quad \int F'/F \cdot dx = \log F.$

903. $\int (F' \cdot f - F \cdot f')/(f)^2 \cdot dx = F/f.$

904. $\int (F' \cdot f - F \cdot f')/Ff \cdot dx = \log (F/f).$

905. $\int \frac{dx}{F \cdot (x^2 - a^2)} = \frac{1}{2a} \int \frac{dx}{F \cdot (x-a)} - \frac{1}{2a} \int \frac{dx}{F \cdot (x+a)}.$

906. $\int \frac{dx}{F(F \pm f)} = \pm \int \frac{dx}{F \cdot f} \mp \int \frac{dx}{f(F \pm f)}.$

907. $\int \frac{F' \cdot dx}{\sqrt{aF+b}} = (2\sqrt{aF+b})/a.$

908. $\int \frac{F' \cdot dx}{\sqrt{F^2+a}} = \log (F + \sqrt{F^2+a}).$

909. $\int \frac{F \cdot dx}{(F+a)(F+b)} = \frac{a}{a-b} \int \frac{dx}{F+a} - \frac{b}{a-b} \int \frac{dx}{F+b}.$

910. $\int \frac{F \cdot dx}{(F+f)^n} = \int \frac{dx}{(F+f)^{n-1}} - \int \frac{f\,dx}{(F+f)^n}.$

911. $\int \frac{F' \cdot dx}{p^2+q^2F^2} = \frac{1}{pq} \cdot \tan^{-1} \frac{qF}{p}, \quad \int \frac{F' \cdot dx}{q^2F^2-p^2} = \frac{1}{2pq} \log \frac{qF-p}{qF+p}.$

912. $\int \frac{F^{2n} \cdot dx}{1 - F^{2n}} = -x + \int \frac{dx}{1 - F^{2n}}.$

913. $\int \frac{F' \cdot dx}{F^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{F}{a}\right).$

914. $\int \frac{F' \cdot dx}{a^2F^2 - b^2} = \frac{1}{2\,ab} \log \frac{aF - b}{aF + b}.$

915. $\int \frac{F^{2n} \cdot dx}{F^{2n} - b^2} = \int \frac{F^n \cdot dx}{2\,(F^n - b)} + \int \frac{F^n \cdot dx}{2\,(F^n + b)}.$

916. $\int \frac{F' \cdot dx}{\sqrt{b^2 - F^2}} = \sin^{-1}\left(\frac{F}{b}\right).$

917. $\int \frac{F' \cdot dx}{aF^2 + bF} = \frac{1}{b} \log \frac{F}{aF + b}.$

918. $\int \frac{F' \cdot dx}{aF^2 - bF} = \frac{1}{b} \log \frac{aF - b}{F}.$

919. $\int \frac{F' \cdot dx}{F\sqrt{F^2 - b^2}} = \frac{1}{b} \sec^{-1}\left(\frac{F}{b}\right).$

920. $\int \frac{(F' \cdot f - F \cdot f')\,dx}{F^2 + f^2} = \tan^{-1}\left(\frac{F}{f}\right).$

921. $\int \frac{(F' \cdot f - F \cdot f')\,dx}{F^2 - f^2} = \frac{1}{2} \log \left(\frac{F - f}{F + f}\right).$

## J. — Integrals Useful in the Theory of Alternating Currents.

922. $\int \sin(\omega t + \phi)\,dt = -\frac{1}{\omega} \cdot \cos(\omega t + \phi).$

923. $\int \cos(\omega t + \phi)\,dt = \frac{1}{\omega} \cdot \sin(\omega t + \phi).$

924. $\int \sin^2(\omega t + \phi)\,dt = \frac{1}{2}t - \frac{1}{4\,\omega} \sin 2(\omega t + \phi).$

925. $$\int \sin(\omega t+\phi)\cdot\cos(\omega t+\phi)\,dt=\frac{1}{2\omega}\cdot\sin^2(\omega t+\phi).$$

926. $$\int \cos^2(\omega t+\phi)\,dt=\frac{1}{2}t+\frac{1}{4\omega}\sin 2(\omega t+\phi).$$

927. $$\int \sin(\omega t+\lambda)\cdot\sin(\omega t+\mu)\,dt=\frac{\cos(\mu-\lambda)}{2\omega}(\omega t) - \frac{\sin(\omega t+\lambda)\cdot\cos(\omega t+\mu)}{2\omega}.$$

928. $$\int \sin(\omega t+\lambda)\cdot\cos(\omega t+\mu)\,dt=\frac{\sin(\omega t+\lambda)\cdot\sin(\omega t+\mu)}{2\omega} - \frac{\sin(\mu-\lambda)}{2\omega}(\omega t).$$

929. $$\int \cos(\omega t+\lambda)\cdot\cos(\omega t+\mu)\,dt=\frac{\cos(\mu-\lambda)}{2\omega}(\omega t) + \frac{\sin(\omega t+\lambda)\cdot\cos(\omega t+\lambda)}{2\omega}.$$

930. $$\int \sin(mt+\lambda)\cdot\sin(nt+\mu)\,dt=\frac{\sin[mt-nt+\lambda-\mu]}{2(m-n)} - \frac{\sin[mt+nt+\lambda+\mu]}{2(m+n)}.$$

931. $$\int \cos(mt+\lambda)\cdot\cos(nt+\mu)\,dt=\frac{\sin[mt+nt+\lambda+\mu]}{2(m+n)} + \frac{\sin[mt-nt+\lambda-\mu]}{2(m-n)}.$$

932. $$\int \sin(mt+\lambda)\cdot\cos(nt+\mu)\,dt=-\frac{\cos[mt+nt+\lambda+\mu]}{2(m+n)} - \frac{\cos[mt-nt+\lambda-\mu]}{2(m-n)}.$$

**933.** $$\int \cos(\omega t+\lambda+mx)\cdot\cos(\omega t+\lambda-mx)\,dx$$
$$=\cos^2(\omega t+\lambda)\left[\frac{mx+\sin mx\cdot\cos mx}{2m}\right]$$
$$-\sin^2(\omega t+\lambda)\left[\frac{mx-\sin mx\cdot\cos mx}{2m}\right].$$

$$\left\{\begin{array}{l} m\cdot\sin(\omega t+\phi)+n\cdot\cos(\omega t+\phi)=\sqrt{m^2+n^2}\cdot\sin(\omega t+\phi+\partial) \\ \text{where } \tan\partial=n/m. \\ m\cdot\sin(\omega t+\phi)-n\cdot\cos(\omega t+\phi)=\sqrt{m^2+n^2}\cdot\sin(\omega t+\phi-\partial). \end{array}\right\}$$

**934.** $$\int e^{(-b\pm ci)t}dt=\frac{-b\mp ci}{b^2+c^2}e^{(-b\pm ci)t}$$
$$=\frac{e^{-bt}}{b^2+c^2}[(c\cdot\sin ct-b\cdot\cos ct)\mp i\,(b\cdot\sin ct+c\cdot\cos ct)]$$
$$=\frac{e^{-bt}}{\sqrt{b^2+c^2}}[\sin(ct-\delta)\mp i\cdot\cos(ct-\delta)],$$
where $\tan\delta=b/c$.

**935.** $$\int e^{\alpha t}\cdot\cos(\omega t+\phi)\,dt$$
$$=\frac{e^{\alpha t}}{\alpha^2+\omega^2}[\omega\sin(\omega t+\phi)+\alpha\cdot\cos(\omega t+\phi)]$$
$$=\frac{e^{\alpha t}}{\sqrt{\alpha^2+\omega^2}}\cos[\omega t+\phi-\tan^{-1}(\omega/\alpha)].$$

**936.** $$\int e^{\alpha t}\cdot\sin(\omega t+\phi)\,dt$$
$$=\frac{e^{\alpha t}}{\alpha^2+\omega^2}[\alpha\cdot\sin(\omega t+\phi)-\omega\cdot\cos(\omega t+\phi)]$$
$$=\frac{e^{\alpha t}}{\sqrt{\alpha^2+\omega^2}}\sin[\omega t+\phi-\tan^{-1}(\omega/\alpha)].$$

**937.** $$\int[e^{\alpha t}\cdot\sin(\omega t+\phi)]^2dt$$
$$=\frac{e^{2\alpha t}}{4}\left[\frac{1}{\alpha}-\frac{\omega\cdot\sin 2(\omega t+\phi)+\alpha\cdot\cos 2(\omega t+\phi)}{\alpha^2+\omega^2}\right]$$
$$=\frac{e^{2\alpha t}}{4}\left[\frac{1}{\alpha}-\frac{\cos[2\omega t+2\phi-\tan^{-1}(\omega/\alpha)]}{\sqrt{\alpha^2+\omega^2}}\right].$$

**938.** $$\int [e^{\alpha t} \cdot \cos(\omega t + \phi)]^2 dt$$

$$= \frac{e^{2\alpha t}}{4}\left[\frac{1}{\alpha} + \frac{\omega \cdot \sin 2(\omega t + \phi) + \alpha \cdot \cos 2(\omega t + \phi)}{\alpha^2 + \omega^2}\right]$$

$$= \frac{e^{2\alpha t}}{4}\left[\frac{1}{\alpha} + \frac{\cos\left[2\omega t + 2\phi - \tan^{-1}(\omega/\alpha)\right]}{\sqrt{\alpha^2 + \omega^2}}\right].$$

---

In the case of a direct trigonometric function of $(\omega t + \phi)$, $T = 2\pi/\omega$ is called the *period* or the *cycle*. The mean value for any whole number of periods, reckoned from any epoch, of $\sin(\omega t + \phi)$, $\cos(\omega t + \phi)$, or $\sin(\omega t + \phi) \cdot \cos(\omega t + \phi)$, is zero, whereas the mean value for any whole number of half periods, reckoned from any epoch, of either $\sin^2(\omega t + \phi)$ or $\cos^2(\omega t + \phi)$ is one half. The mean value of $\sin(\omega t)$ from $t = 0$ to $t = \frac{1}{2} T$, or of $\cos(\omega t)$ from $-\frac{1}{4} T$ to $+\frac{1}{4} T$, is $2/\pi$ or 0.6366.

The mean value, for any number of whole periods, of either $\sin(\omega t + \lambda) \cdot \sin(\omega t + \mu)$ or $\cos(\omega t + \lambda) \cdot \cos(\omega t + \mu)$ is $\frac{1}{2} \cdot \cos(\lambda - \mu)$, while the mean value of $\sin(\omega t + \lambda) \cdot \cos(\omega t + \mu)$ is $\frac{1}{2} \sin(\lambda - \mu)$.

## INTERPOLATION.

If values of an analytic function, $f(x)$, are given in a table for a number of values of the argument $x$, separated from one another consecutively by the constant small interval, $\delta$, the differences between successive tabular values of the function are called *first tabular differences*, the differences of these first differences, *second tabular differences*, and so on. The tabular differences of the first, second, third, and fourth orders corresponding to $x = a$ are

$$\Delta_1 \equiv f(a+\delta) - f(a),$$
$$\Delta_2 \equiv f(a+2\delta) - 2 \cdot f(a+\delta) + f(a),$$
$$\Delta_3 \equiv f(a+3\delta) - 3 \cdot f(a+2\delta) + 3 \cdot f(a+\delta) - f(a),$$
$$\Delta_4 \equiv f(a+4\delta) - 4 \cdot f(a+3\delta) + 6 \cdot f(a+2\delta) - 4 \cdot f(a+\delta) + f(a),$$

where $f(a)$ is any tabulated value.

The value of the function for $x = (a+h)$, where $h = k\delta$, is

$$f(a+h) = f(a) + k \cdot \Delta_1 + \frac{k(k-1)}{2!} \cdot \Delta_2 + \frac{k(k-1)(k-2)}{3!} \cdot \Delta_3 + \frac{k(k-1)(k-2)(k-3)}{4!} \cdot \Delta_4 + \cdots.$$

## The Probability Integral.

$$\left(\frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}dx.\right)$$

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.00 | 0.00000 | 00113 | 00226 | 00339 | 00451 | 00564 | 00677 | 00790 | 00903 | 01016 |
| 0.01 | 0.01128 | 01241 | 01354 | 01467 | 01580 | 01692 | 01805 | 01918 | 02031 | 02144 |
| 0.02 | 0.02256 | 02369 | 02482 | 02595 | 02708 | 02820 | 02933 | 03046 | 03159 | 03271 |
| 0.03 | 0.03384 | 03497 | 03610 | 03722 | 03835 | 03948 | 04060 | 04173 | 04286 | 04398 |
| 0.04 | 0.04511 | 04624 | 04736 | 04849 | 04962 | 05074 | 05187 | 05299 | 05412 | 05525 |
| 0.05 | 0.05637 | 05750 | 05862 | 05975 | 06087 | 06200 | 06312 | 06425 | 06537 | 06650 |
| 0.06 | 0.06762 | 06875 | 06987 | 07099 | 07212 | 07324 | 07437 | 07549 | 07661 | 07773 |
| 0.07 | 0.07886 | 07998 | 08110 | 08223 | 08335 | 08447 | 08559 | 08671 | 08784 | 08896 |
| 0.08 | 0.09008 | 09120 | 09232 | 09344 | 09456 | 09568 | 09680 | 09792 | 09904 | 10016 |
| 0.09 | 0.10128 | 10240 | 10352 | 10464 | 10576 | 10687 | 10799 | 10911 | 11023 | 11135 |
| 0.10 | 0.11246 | 11358 | 11470 | 11581 | 11693 | 11805 | 11916 | 12028 | 12139 | 12251 |
| 0.11 | 0.12362 | 12474 | 12585 | 12697 | 12808 | 12919 | 13031 | 13142 | 13253 | 13365 |
| 0.12 | 0.13476 | 13587 | 13698 | 13809 | 13921 | 14032 | 14143 | 14254 | 14365 | 14476 |
| 0.13 | 0.14587 | 14698 | 14809 | 14919 | 15030 | 15141 | 15252 | 15363 | 15473 | 15584 |
| 0.14 | 0.15695 | 15805 | 15916 | 16027 | 16137 | 16248 | 16358 | 16468 | 16579 | 16689 |
| 0.15 | 0.16800 | 16910 | 17020 | 17130 | 17241 | 17351 | 17461 | 17571 | 17681 | 17791 |
| 0.16 | 0.17901 | 18011 | 18121 | 18231 | 18341 | 18451 | 18560 | 18670 | 18780 | 18890 |
| 0.17 | 0.18999 | 19109 | 19218 | 19328 | 19437 | 19547 | 19656 | 19766 | 19875 | 19984 |
| 0.18 | 0.20094 | 20203 | 20312 | 20421 | 20530 | 20639 | 20748 | 20857 | 20966 | 21075 |
| 0.19 | 0.21184 | 21293 | 21402 | 21510 | 21619 | 21728 | 21836 | 21945 | 22053 | 22162 |
| 0.20 | 0.22270 | 22379 | 22487 | 22595 | 22704 | 22812 | 22920 | 23028 | 23136 | 23244 |
| 0.21 | 0.23352 | 23460 | 23568 | 23676 | 23784 | 23891 | 23999 | 24107 | 24214 | 24322 |
| 0.22 | 0.24430 | 24537 | 24645 | 24752 | 24859 | 24967 | 25074 | 25181 | 25288 | 25395 |
| 0.23 | 0.25502 | 25609 | 25716 | 25823 | 25930 | 26037 | 26144 | 26250 | 26357 | 26463 |
| 0.24 | 0.26570 | 26677 | 26783 | 26889 | 26996 | 27102 | 27208 | 27314 | 27421 | 27527 |
| 0.25 | 0.27633 | 27739 | 27845 | 27950 | 28056 | 28162 | 28268 | 28373 | 28479 | 28584 |
| 0.26 | 0.28690 | 28795 | 28901 | 29006 | 29111 | 29217 | 29322 | 29427 | 29532 | 29637 |
| 0.27 | 0.29742 | 29847 | 29952 | 30056 | 30161 | 30266 | 30370 | 30475 | 30579 | 30684 |
| 0.28 | 0.30788 | 30892 | 30997 | 31101 | 31205 | 31309 | 31413 | 31517 | 31621 | 31725 |
| 0.29 | 0.31828 | 31922 | 32036 | 32139 | 32243 | 32346 | 32450 | 32553 | 32656 | 32760 |
| 0.30 | 0.32863 | 32966 | 33069 | 33172 | 33275 | 33378 | 33480 | 33583 | 33686 | 33788 |
| 0.31 | 0.33891 | 33993 | 34096 | 34198 | 34300 | 34403 | 34505 | 34607 | 34709 | 34811 |
| 0.32 | 0.34913 | 35014 | 35116 | 35218 | 35319 | 35421 | 35523 | 35624 | 35725 | 35827 |
| 0.33 | 0.35928 | 36029 | 36130 | 36231 | 36332 | 36433 | 36534 | 36635 | 36735 | 36836 |
| 0.34 | 0.36936 | 37037 | 37137 | 37238 | 37338 | 37438 | 37538 | 37638 | 37738 | 37838 |
| 0.35 | 0.37938 | 38038 | 38138 | 38237 | 38337 | 38436 | 38536 | 38635 | 38735 | 38834 |
| 0.36 | 0.38933 | 39032 | 39131 | 39230 | 39329 | 39428 | 39526 | 39625 | 39724 | 39822 |
| 0.37 | 0.39921 | 40019 | 40117 | 40215 | 40314 | 40412 | 40510 | 40608 | 40705 | 40803 |
| 0.38 | 0.40901 | 40999 | 41096 | 41194 | 41291 | 41388 | 41486 | 41583 | 41680 | 41777 |
| 0.39 | 0.41874 | 41971 | 42068 | 42164 | 42261 | 42358 | 42454 | 42550 | 42647 | 42743 |
| 0.40 | 0.42839 | 42935 | 43031 | 43127 | 43223 | 43319 | 43415 | 43510 | 43606 | 43701 |
| 0.41 | 0.43797 | 43892 | 43988 | 44083 | 44178 | 44273 | 44368 | 44463 | 44557 | 44652 |
| 0.42 | 0.44747 | 44841 | 44936 | 45030 | 45124 | 45219 | 45313 | 45407 | 45501 | 45595 |
| 0.43 | 0.45689 | 45782 | 45876 | 45970 | 46063 | 46157 | 46250 | 46343 | 46436 | 46529 |
| 0.44 | 0.46623 | 46715 | 46808 | 46901 | 46994 | 47086 | 47179 | 47271 | 47364 | 47456 |
| 0.45 | 0.47548 | 47640 | 47732 | 47824 | 47916 | 48008 | 48100 | 48191 | 48283 | 48374 |
| 0.46 | 0.48466 | 48557 | 48648 | 48739 | 48830 | 48921 | 49012 | 49103 | 49193 | 49284 |
| 0.47 | 0.49375 | 49465 | 49555 | 49646 | 49736 | 49826 | 49916 | 50006 | 50096 | 50185 |
| 0.48 | 0.50275 | 50365 | 50454 | 50543 | 50633 | 50722 | 50811 | 50900 | 50989 | 51078 |
| 0.49 | 0.51167 | 51256 | 51344 | 51433 | 51521 | 51609 | 51698 | 51786 | 51874 | 51962 |

## The Probability Integral.

$$\left(\frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}dx.\right)$$

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.50 | 0.52050 | 52138 | 52226 | 52313 | 52401 | 52488 | 52576 | 52663 | 52750 | 52837 |
| 0.51 | 0.52924 | 53011 | 53098 | 53185 | 53272 | 53358 | 53445 | 53531 | 53617 | 53704 |
| 0.52 | 0.53790 | 53876 | 53962 | 54048 | 54134 | 54219 | 54305 | 54390 | 54476 | 54561 |
| 0.53 | 0.54646 | 54732 | 54817 | 54902 | 54987 | 55071 | 55156 | 55241 | 55325 | 55410 |
| 0.54 | 0.55494 | 55578 | 55662 | 55746 | 55830 | 55914 | 55998 | 56082 | 56165 | 56249 |
| 0.55 | 0.56332 | 56416 | 56499 | 56582 | 56665 | 56748 | 56831 | 56914 | 56996 | 57079 |
| 0.56 | 0.57162 | 57244 | 57326 | 57409 | 57491 | 57573 | 57655 | 57737 | 57818 | 57900 |
| 0.57 | 0.57982 | 58063 | 58144 | 58226 | 58307 | 58388 | 58469 | 58550 | 58631 | 58712 |
| 0.58 | 0.58792 | 58873 | 58953 | 59034 | 59114 | 59194 | 59274 | 59354 | 59434 | 59514 |
| 0.59 | 0.59594 | 59673 | 59753 | 59832 | 59912 | 59991 | 60070 | 60149 | 60228 | 60307 |
| 0.60 | 0.60386 | 60464 | 60543 | 60621 | 60700 | 60778 | 60856 | 60934 | 61012 | 61090 |
| 0.61 | 0.61168 | 61246 | 61323 | 61401 | 61478 | 61556 | 61633 | 61710 | 61787 | 61864 |
| 0.62 | 0.61941 | 62018 | 62095 | 62171 | 62248 | 62324 | 62400 | 62477 | 62553 | 62629 |
| 0.63 | 0.62705 | 62780 | 62856 | 62932 | 63007 | 63083 | 63158 | 63233 | 63309 | 63384 |
| 0.64 | 0.63459 | 63533 | 63608 | 63683 | 63757 | 63832 | 63906 | 63981 | 64055 | 64129 |
| 0.65 | 0.64203 | 64277 | 64351 | 64424 | 64498 | 64572 | 64645 | 64718 | 64791 | 64865 |
| 0.66 | 0.64938 | 65011 | 65083 | 65156 | 65229 | 65301 | 65374 | 65446 | 65519 | 65591 |
| 0.67 | 0.65663 | 65735 | 65807 | 65878 | 65950 | 66022 | 66093 | 66165 | 66236 | 66307 |
| 0.68 | 0.66378 | 66449 | 66520 | 66591 | 66662 | 66732 | 66803 | 66873 | 66944 | 67014 |
| 0.69 | 0.67084 | 67154 | 67224 | 67294 | 67364 | 67433 | 67503 | 67572 | 67642 | 67711 |
| 0.70 | 0.67780 | 67849 | 67918 | 67987 | 68056 | 68125 | 68193 | 68262 | 68330 | 68398 |
| 0.71 | 0.68467 | 68535 | 68603 | 68671 | 68738 | 68806 | 68874 | 68941 | 69009 | 69076 |
| 0.72 | 0.69143 | 69210 | 69278 | 69344 | 69411 | 69478 | 69545 | 69611 | 69678 | 69744 |
| 0.73 | 0.69810 | 69877 | 69943 | 70009 | 70075 | 70140 | 70206 | 70272 | 70337 | 70403 |
| 0.74 | 0.70468 | 70533 | 70598 | 70663 | 70728 | 70793 | 70858 | 70922 | 70987 | 71051 |
| 0.75 | 0.71116 | 71180 | 71244 | 71308 | 71372 | 71436 | 71500 | 71563 | 71627 | 71690 |
| 0.76 | 0.71754 | 71817 | 71880 | 71943 | 72006 | 72069 | 72132 | 72195 | 72257 | 72320 |
| 0.77 | 0.72382 | 72444 | 72507 | 72569 | 72631 | 72693 | 72755 | 72816 | 72878 | 72940 |
| 0.78 | 0.73001 | 73062 | 73124 | 73185 | 73246 | 73307 | 73368 | 73429 | 73489 | 73550 |
| 0.79 | 0.73610 | 73671 | 73731 | 73791 | 73851 | 73911 | 73971 | 74031 | 74091 | 74151 |
| 0.80 | 0.74210 | 74270 | 74329 | 74388 | 74447 | 74506 | 74565 | 74624 | 74683 | 74742 |
| 0.81 | 0.74800 | 74859 | 74917 | 74976 | 75034 | 75092 | 75150 | 75208 | 75266 | 75323 |
| 0.82 | 0.75381 | 75439 | 75496 | 75553 | 75611 | 75668 | 75725 | 75782 | 75839 | 75896 |
| 0.83 | 0.75952 | 76009 | 76066 | 76122 | 76178 | 76234 | 76291 | 76347 | 76403 | 76459 |
| 0.84 | 0.76514 | 76570 | 76626 | 76681 | 76736 | 76792 | 76847 | 76902 | 76957 | 77012 |
| 0.85 | 0.77067 | 77122 | 77176 | 77231 | 77285 | 77340 | 77394 | 77448 | 77502 | 77556 |
| 0.86 | 0.77610 | 77664 | 77718 | 77771 | 77825 | 77878 | 77932 | 77985 | 78038 | 78091 |
| 0.87 | 0.78144 | 78197 | 78250 | 78302 | 78355 | 78408 | 78460 | 78512 | 78565 | 78617 |
| 0.88 | 0.78669 | 78721 | 78773 | 78824 | 78876 | 78928 | 78979 | 79031 | 79082 | 79133 |
| 0.89 | 0.79184 | 79235 | 79286 | 79337 | 79388 | 79439 | 79489 | 79540 | 79590 | 79641 |
| 0.90 | 0.79691 | 79741 | 79791 | 79841 | 79891 | 79941 | 79990 | 80040 | 80090 | 80139 |
| 0.91 | 0.80188 | 80238 | 80287 | 80336 | 80385 | 80434 | 80482 | 80531 | 80580 | 80628 |
| 0.92 | 0.80677 | 80725 | 80773 | 80822 | 80870 | 80918 | 80966 | 81013 | 81061 | 81109 |
| 0.93 | 0.81156 | 81204 | 81251 | 81299 | 81346 | 81393 | 81440 | 81487 | 81534 | 81580 |
| 0.94 | 0.81627 | 81674 | 81720 | 81767 | 81813 | 81859 | 81905 | 81951 | 81997 | 82043 |
| 0.95 | 0.82089 | 82135 | 82180 | 82226 | 82271 | 82317 | 82362 | 82407 | 82452 | 82497 |
| 0.96 | 0.82542 | 82587 | 82632 | 82677 | 82721 | 82766 | 82810 | 82855 | 82899 | 82943 |
| 0.97 | 0.82987 | 83031 | 83075 | 83119 | 83162 | 83206 | 83250 | 83293 | 83337 | 83380 |
| 0.98 | 0.83423 | 83466 | 83509 | 83552 | 83595 | 83638 | 83681 | 83723 | 83766 | 83808 |
| 0.99 | 0.83851 | 83893 | 83935 | 83977 | 84020 | 84061 | 84103 | 84145 | 84187 | 84229 |

## The Probability Integral.

$$\left(\frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}\,dx.\right)$$

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1.00 | 0.84270 | 84312 | 84353 | 84394 | 84435 | 84477 | 84518 | 84559 | 84600 | 84640 |
| 1.01 | 0.84681 | 84722 | 84762 | 84803 | 84843 | 84883 | 84924 | 84964 | 85004 | 85044 |
| 1.02 | 0.85084 | 85124 | 85163 | 85203 | 85243 | 85282 | 85322 | 85361 | 85400 | 85439 |
| 1.03 | 0.85478 | 85517 | 85556 | 85595 | 85634 | 85673 | 85711 | 85750 | 85788 | 85827 |
| 1.04 | 0.85865 | 85903 | 85941 | 85979 | 86017 | 86055 | 86093 | 86131 | 86169 | 86206 |
| 1.05 | 0.86244 | 86281 | 86318 | 86356 | 86393 | 86430 | 86467 | 86504 | 86541 | 86578 |
| 1.06 | 0.86614 | 86651 | 86688 | 86724 | 86760 | 86797 | 86833 | 86869 | 86905 | 86941 |
| 1.07 | 0.86977 | 87013 | 87049 | 87085 | 87120 | 87156 | 87191 | 87227 | 87262 | 87297 |
| 1.08 | 0.87333 | 87368 | 87403 | 87438 | 87473 | 87507 | 87542 | 87577 | 87611 | 87646 |
| 1.09 | 0.87680 | 87715 | 87749 | 87783 | 87817 | 87851 | 87885 | 87919 | 87953 | 87987 |
| 1.10 | 0.88021 | 88054 | 88088 | 88121 | 88155 | 88188 | 88221 | 88254 | 88287 | 88320 |
| 1.11 | 0.88353 | 88386 | 88419 | 88452 | 88484 | 88517 | 88549 | 88582 | 88614 | 88647 |
| 1.12 | 0.88679 | 88711 | 88743 | 88775 | 88807 | 88839 | 88871 | 88902 | 88934 | 88966 |
| 1.13 | 0.88997 | 89029 | 89060 | 89091 | 89122 | 89154 | 89185 | 89216 | 89247 | 89277 |
| 1.14 | 0.89308 | 89339 | 89370 | 89400 | 89431 | 89461 | 89492 | 89522 | 89552 | 89582 |
| 1.15 | 0.89612 | 89642 | 89672 | 89702 | 89732 | 89762 | 89792 | 89821 | 89851 | 89880 |
| 1.16 | 0.89910 | 89939 | 89968 | 89997 | 90027 | 90056 | 90085 | 90114 | 90142 | 90171 |
| 1.17 | 0.90200 | 90229 | 90257 | 90286 | 90314 | 90343 | 90371 | 90399 | 90428 | 90456 |
| 1.18 | 0.90484 | 90512 | 90540 | 90568 | 90595 | 90623 | 90651 | 90678 | 90706 | 90733 |
| 1.19 | 0.90761 | 90788 | 90815 | 90843 | 90870 | 90897 | 90924 | 90951 | 90978 | 91005 |
| 1.20 | 0.91031 | 91058 | 91085 | 91111 | 91138 | 91164 | 91191 | 91217 | 91243 | 91269 |
| 1.21 | 0.91296 | 91322 | 91348 | 91374 | 91399 | 91425 | 91451 | 91477 | 91502 | 91528 |
| 1.22 | 0.91553 | 91579 | 91604 | 91630 | 91655 | 91680 | 91705 | 91730 | 91755 | 91780 |
| 1.23 | 0.91805 | 91830 | 91855 | 91879 | 91904 | 91929 | 91953 | 91978 | 92002 | 92026 |
| 1.24 | 0.92051 | 92075 | 92099 | 92123 | 92147 | 92171 | 92195 | 92219 | 92243 | 92266 |
| 1.25 | 0.92290 | 92314 | 92337 | 92361 | 92384 | 92408 | 92431 | 92454 | 92477 | 92500 |
| 1.26 | 0.92524 | 92547 | 92570 | 92593 | 92615 | 92638 | 92661 | 92684 | 92706 | 92729 |
| 1.27 | 0.92751 | 92774 | 92796 | 92819 | 92841 | 92863 | 92885 | 92907 | 92929 | 92951 |
| 1.28 | 0.92973 | 92995 | 93017 | 93039 | 93061 | 93082 | 93104 | 93126 | 93147 | 93168 |
| 1.29 | 0.93190 | 93211 | 93232 | 93254 | 93275 | 93296 | 93317 | 93338 | 93359 | 93380 |
| 1.30 | 0.93401 | 93422 | 93442 | 93463 | 93484 | 93504 | 93525 | 93545 | 93566 | 93586 |
| 1.31 | 0.93606 | 93627 | 93647 | 93667 | 93687 | 93707 | 93727 | 93747 | 93767 | 93787 |
| 1.32 | 0.93807 | 93826 | 93846 | 93866 | 93885 | 93905 | 93924 | 93944 | 93963 | 93982 |
| 1.33 | 0.94002 | 94021 | 94040 | 94059 | 94078 | 94097 | 94116 | 94135 | 94154 | 94173 |
| 1.34 | 0.94191 | 94210 | 94229 | 94247 | 94266 | 94284 | 94303 | 94321 | 94340 | 94358 |
| 1.35 | 0.94376 | 94394 | 94413 | 94431 | 94449 | 94467 | 94485 | 94503 | 94521 | 94538 |
| 1.36 | 0.94556 | 94574 | 94592 | 94609 | 94627 | 94644 | 94662 | 94679 | 94697 | 94714 |
| 1.37 | 0.94731 | 94748 | 94766 | 94783 | 94800 | 94817 | 94834 | 94851 | 94868 | 94885 |
| 1.38 | 0.94902 | 94918 | 94935 | 94952 | 94968 | 94985 | 95002 | 95018 | 95035 | 95051 |
| 1.39 | 0.95067 | 95084 | 95100 | 95116 | 95132 | 95148 | 95165 | 95181 | 95197 | 95213 |
| 1.40 | 0.95229 | 95244 | 95260 | 95276 | 95292 | 95307 | 95323 | 95339 | 95354 | 95370 |
| 1.41 | 0.95385 | 95401 | 95416 | 95431 | 95447 | 95462 | 95477 | 95492 | 95507 | 95523 |
| 1.42 | 0.95538 | 95553 | 95568 | 95582 | 95597 | 95612 | 95627 | 95642 | 95656 | 95671 |
| 1.43 | 0.95686 | 95700 | 95715 | 95729 | 95744 | 95758 | 95773 | 95787 | 95801 | 95815 |
| 1.44 | 0.95830 | 95844 | 95858 | 95872 | 95886 | 95900 | 95914 | 95928 | 95942 | 95956 |
| 1.45 | 0.95970 | 95983 | 95997 | 96011 | 96024 | 96038 | 96051 | 96065 | 96078 | 96092 |
| 1.46 | 0.96105 | 96119 | 96132 | 96145 | 96159 | 96172 | 96185 | 96198 | 96211 | 96224 |
| 1.47 | 0.96237 | 96250 | 96263 | 96276 | 96289 | 96302 | 96315 | 96327 | 96340 | 96353 |
| 1.48 | 0.96365 | 96378 | 96391 | 96403 | 96416 | 96428 | 96440 | 96453 | 96465 | 96478 |
| 1.49 | 0.96490 | 96502 | 96514 | 96526 | 96539 | 96551 | 96563 | 96575 | 96587 | 96599 |

## The Probability Integral.

$$\left(\frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}dx.\right)$$

| $x$ | 0 | 2 | 4 | 6 | 8 | $x$ | 0 | 2 | 4 | 6 | 8 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1.50 | 0.96611 | 96634 | 96658 | 96681 | 96705 | 2.00 | 0.99532 | 99536 | 99540 | 99544 | 99548 |
| 1.51 | 0.96728 | 96751 | 96774 | 96796 | 96819 | 2.01 | 0.99552 | 99556 | 99560 | 99564 | 99568 |
| 1.52 | 0.96841 | 96864 | 96886 | 96908 | 96930 | 2.02 | 0.99572 | 99576 | 99580 | 99583 | 99587 |
| 1.53 | 0.96952 | 96973 | 96995 | 97016 | 97037 | 2.03 | 0.99591 | 99594 | 99598 | 99601 | 99605 |
| 1.54 | 0.97059 | 97080 | 97100 | 97121 | 97142 | 2.04 | 0.99609 | 99612 | 99616 | 99619 | 99622 |
| 1.55 | 0.97162 | 97183 | 97203 | 97223 | 97243 | 2.05 | 0.99626 | 99629 | 99633 | 99636 | 99639 |
| 1.56 | 0.97263 | 97283 | 97302 | 97322 | 97341 | 2.06 | 0.99642 | 99646 | 99649 | 99652 | 99655 |
| 1.57 | 0.97360 | 97379 | 97398 | 97417 | 97436 | 2.07 | 0.99658 | 99661 | 99664 | 99667 | 99670 |
| 1.58 | 0.97455 | 97473 | 97492 | 97510 | 97528 | 2.08 | 0.99673 | 99676 | 99679 | 99682 | 99685 |
| 1.59 | 0.97546 | 97564 | 97582 | 97600 | 97617 | 2.09 | 0.99688 | 99691 | 99694 | 99697 | 99699 |
| 1.60 | 0.97635 | 97652 | 97670 | 97687 | 97704 | 2.10 | 0.99702 | 99705 | 99707 | 99710 | 99713 |
| 1.61 | 0.97721 | 97738 | 97754 | 97771 | 97787 | 2.11 | 0.99715 | 99718 | 99721 | 99723 | 99726 |
| 1.62 | 0.97804 | 97820 | 97836 | 97852 | 97868 | 2.12 | 0.99728 | 99731 | 99733 | 99736 | 99738 |
| 1.63 | 0.97884 | 97900 | 97916 | 97931 | 97947 | 2.13 | 0.99741 | 99743 | 99745 | 99748 | 99750 |
| 1.64 | 0.97962 | 97977 | 97993 | 98008 | 98023 | 2.14 | 0.99753 | 99755 | 99757 | 99759 | 99762 |
| 1.65 | 0.98038 | 98052 | 98067 | 98082 | 98096 | 2.15 | 0.99764 | 99766 | 99768 | 99770 | 99773 |
| 1.66 | 0.98110 | 98125 | 98139 | 98153 | 98167 | 2.16 | 0.99775 | 99777 | 99779 | 99781 | 99783 |
| 1.67 | 0.98181 | 98195 | 98209 | 98222 | 98236 | 2.17 | 0.99785 | 99787 | 99789 | 99791 | 99793 |
| 1.68 | 0.98249 | 98263 | 98276 | 98289 | 98302 | 2.18 | 0.99795 | 99797 | 99799 | 99801 | 99803 |
| 1.69 | 0.98315 | 98328 | 98341 | 98354 | 98366 | 2.19 | 0.99805 | 99806 | 99808 | 99810 | 99812 |
| 1.70 | 0.98379 | 98392 | 98404 | 98416 | 98429 | 2.20 | 0.99814 | 99815 | 99817 | 99819 | 99821 |
| 1.71 | 0.98441 | 98453 | 98465 | 98477 | 98489 | 2.21 | 0.99822 | 99824 | 99826 | 99827 | 99829 |
| 1.72 | 0.98500 | 98512 | 98524 | 98535 | 98546 | 2.22 | 0.99831 | 99832 | 99834 | 99836 | 99837 |
| 1.73 | 0.98558 | 98569 | 98580 | 98591 | 98602 | 2.23 | 0.99839 | 99840 | 99842 | 99843 | 99845 |
| 1.74 | 0.98613 | 98624 | 98635 | 98646 | 98657 | 2.24 | 0.99846 | 99848 | 99849 | 99851 | 99852 |
| 1.75 | 0.98667 | 98678 | 98688 | 98699 | 98709 | 2.25 | 0.99854 | 99855 | 99857 | 99858 | 99859 |
| 1.76 | 0.98719 | 98729 | 98739 | 98749 | 98759 | 2.26 | 0.99861 | 99862 | 99863 | 99865 | 99866 |
| 1.77 | 0.98769 | 98779 | 98789 | 98798 | 98808 | 2.27 | 0.99867 | 99869 | 99870 | 99871 | 99873 |
| 1.78 | 0.98817 | 98827 | 98836 | 98846 | 98855 | 2.28 | 0.99874 | 99875 | 99876 | 99877 | 99879 |
| 1.79 | 0.98864 | 98873 | 98882 | 98891 | 98900 | 2.29 | 0.99880 | 99881 | 99882 | 99883 | 99885 |
| 1.80 | 0.98909 | 98918 | 98927 | 98935 | 98944 | 2.30 | 0.99886 | 99887 | 99888 | 99889 | 99890 |
| 1.81 | 0.98952 | 98961 | 98969 | 98978 | 98986 | 2.31 | 0.99891 | 99892 | 99893 | 99894 | 99896 |
| 1.82 | 0.98994 | 99003 | 99011 | 99019 | 99027 | 2.32 | 0.99897 | 99898 | 99899 | 99900 | 99901 |
| 1.83 | 0.99035 | 99043 | 99050 | 99058 | 99066 | 2.33 | 0.99902 | 99903 | 99904 | 99905 | 99906 |
| 1.84 | 0.99074 | 99081 | 99089 | 99096 | 99104 | 2.34 | 0.99906 | 99907 | 99908 | 99909 | 99910 |
| 1.85 | 0.99111 | 99118 | 99126 | 99133 | 99140 | 2.35 | 0.99911 | 99912 | 99913 | 99914 | 99915 |
| 1.86 | 0.99147 | 99154 | 99161 | 99168 | 99175 | 2.36 | 0.99915 | 99916 | 99917 | 99918 | 99919 |
| 1.87 | 0.99182 | 99189 | 99196 | 99202 | 99209 | 2.37 | 0.99920 | 99920 | 99921 | 99922 | 99923 |
| 1.88 | 0.99216 | 99222 | 99229 | 99235 | 99242 | 2.38 | 0.99924 | 99924 | 99925 | 99926 | 99927 |
| 1.89 | 0.99248 | 99254 | 99261 | 99267 | 99273 | 2.39 | 0.99928 | 99928 | 99929 | 99930 | 99930 |
| 1.90 | 0.99279 | 99285 | 99291 | 99297 | 99303 | 2.40 | 0.99931 | 99932 | 99933 | 99933 | 99934 |
| 1.91 | 0.99309 | 99315 | 99321 | 99326 | 99332 | 2.41 | 0.99935 | 99935 | 99936 | 99937 | 99937 |
| 1.92 | 0.99338 | 99343 | 99349 | 99355 | 99360 | 2.42 | 0.99938 | 99939 | 99939 | 99940 | 99940 |
| 1.93 | 0.99366 | 99371 | 99376 | 99382 | 99387 | 2.43 | 0.99941 | 99942 | 99942 | 99943 | 99943 |
| 1.94 | 0.99392 | 99397 | 99403 | 99408 | 99413 | 2.44 | 0.99944 | 99945 | 99945 | 99946 | 99946 |
| 1.95 | 0.99418 | 99423 | 99428 | 99433 | 99438 | 2.45 | 0.99947 | 99947 | 99948 | 99949 | 99949 |
| 1.96 | 0.99443 | 99447 | 99452 | 99457 | 99462 | 2.46 | 0.99950 | 99950 | 99951 | 99951 | 99952 |
| 1.97 | 0.99466 | 99471 | 99476 | 99480 | 99485 | 2.47 | 0.99952 | 99953 | 99953 | 99954 | 99954 |
| 1.98 | 0.99489 | 99494 | 99498 | 99502 | 99507 | 2.48 | 0.99955 | 99955 | 99956 | 99956 | 99957 |
| 1.99 | 0.99511 | 99515 | 99520 | 99524 | 99528 | 2.49 | 0.99957 | 99958 | 99958 | 99958 | 99959 |
| 2.00 | 0.99532 | 99536 | 99540 | 99544 | 99548 | 2.50 | 0.99959 | 99960 | 99960 | 99961 | 99961 |

## The Probability Integral.

$$\left(\frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}dx.\right)$$

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 2.5 | 0.99959 | 99961 | 99963 | 99965 | 99967 | 99969 | 99971 | 99972 | 99974 | 99975 |
| 2.6 | 0.99976 | 99978 | 99979 | 99980 | 99981 | 99982 | 99983 | 99984 | 99985 | 99986 |
| 2.7 | 0.99987 | 99987 | 99988 | 99989 | 99989 | 99990 | 99991 | 99991 | 99992 | 99992 |
| 2.8 | 0.99992 | 99993 | 99993 | 99994 | 99994 | 99994 | 99995 | 99995 | 99995 | 99996 |
| 2.9 | 0.99996 | 99996 | 99996 | 99997 | 99997 | 99997 | 99997 | 99997 | 99997 | 99998 |
| 3.0 | 0.99998 | 99998 | 99998 | 99998 | 99998 | 99998 | 99998 | 99998 | 99999 | 99999 |

The value, $I$, of the Probability Integral may always be found from the convergent series

$$I=\frac{2}{\sqrt{\pi}}\left(x-\frac{x^3}{3\cdot 1!}+\frac{x^5}{5\cdot 2!}-\frac{x^7}{7\cdot 3!}+\cdots\right),$$

but for large values of $x$, the semiconvergent series

$$I=1-\frac{e^{-x^2}}{x\sqrt{\pi}}\left(1-\frac{1}{2x^2}+\frac{1\cdot 3}{(2x^2)^2}-\frac{1\cdot 3\cdot 5}{(2x^2)^3}+\cdots\right)$$

is convenient

**Values of the Complete Elliptic Integrals, $K$ and $E$, for Different Values of the Modulus, $k$.**

$$K = \int_0^{\frac{\pi}{2}} \frac{dz}{\sqrt{1 - k^2 \sin^2 z}}; \quad E = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 z} \cdot dz.$$

| $\sin^{-1} k$ | $K$ | $E$ | $\sin^{-1} k$ | $K$ | $E$ | $\sin^{-1} k$ | $K$ | $E$ |
|---|---|---|---|---|---|---|---|---|
| 0° | 1.5708 | 1.5708 | 30° | 1.6858 | 1.4675 | 60° | 2.1565 | 1.2111 |
| 1° | 1.5709 | 1.5707 | 31° | 1.6941 | 1.4608 | 61° | 2.1842 | 1.2015 |
| 2° | 1.5713 | 1.5703 | 32° | 1.7028 | 1.4539 | 62° | 2.2132 | 1.1920 |
| 3° | 1.5719 | 1.5697 | 33° | 1.7119 | 1.4469 | 63° | 2.2435 | 1.1826 |
| 4° | 1.5727 | 1.5689 | 34° | 1.7214 | 1.4397 | 64° | 2.2754 | 1.1732 |
| 5° | 1.5738 | 1.5678 | 35° | 1.7312 | 1.4323 | 65° | 2.3088 | 1.1638 |
| 6° | 1.5751 | 1.5665 | 36° | 1.7415 | 1.4248 | 66° | 2.3439 | 1.1545 |
| 7° | 1.5767 | 1.5649 | 37° | 1.7522 | 1.4171 | 67° | 2.3809 | 1.1453 |
| 8° | 1.5785 | 1.5632 | 38° | 1.7633 | 1.4092 | 68° | 2.4198 | 1.1362 |
| 9° | 1.5805 | 1.5611 | 39° | 1.7748 | 1.4013 | 69° | 2.4610 | 1.1272 |
| 10° | 1.5828 | 1.5589 | 40° | 1.7868 | 1.3931 | 70° | 2.5046 | 1.1184 |
| 11° | 1.5854 | 1.5564 | 41° | 1.7992 | 1.3849 | 71° | 2.5507 | 1.1096 |
| 12° | 1.5882 | 1.5537 | 42° | 1.8122 | 1.3765 | 72° | 2.5998 | 1.1011 |
| 13° | 1.5913 | 1.5507 | 43° | 1.8256 | 1.3680 | 73° | 2.6521 | 1.0927 |
| 14° | 1.5946 | 1.5476 | 44° | 1.8396 | 1.3594 | 74° | 2.7081 | 1.0844 |
| 15° | 1.5981 | 1.5442 | 45° | 1.8541 | 1.3506 | 75° | 2.7681 | 1.0764 |
| 16° | 1.6020 | 1.5405 | 46° | 1.8691 | 1.3418 | 76° | 2.8327 | 1.0686 |
| 17° | 1.6061 | 1.5367 | 47° | 1.8848 | 1.3329 | 77° | 2.9026 | 1.0611 |
| 18° | 1.6105 | 1.5326 | 48° | 1.9011 | 1.3238 | 78° | 2.9786 | 1.0538 |
| 19° | 1.6151 | 1.5283 | 49° | 1.9180 | 1.3147 | 79° | 3.0617 | 1.0468 |
| 20° | 1.6200 | 1.5238 | 50° | 1.9356 | 1.3055 | 80° | 3.1534 | 1.0401 |
| 21° | 1.6252 | 1.5191 | 51° | 1.9539 | 1.2963 | 81° | 3.2553 | 1.0338 |
| 22° | 1.6307 | 1.5141 | 52° | 1.9729 | 1.2870 | 82° | 3.3699 | 1.0278 |
| 23° | 1.6365 | 1.5090 | 53° | 1.9927 | 1.2776 | 83° | 3.5004 | 1.0223 |
| 24° | 1.6426 | 1.5037 | 54° | 2.0133 | 1.2681 | 84° | 3.6519 | 1.0172 |
| 25° | 1.6490 | 1.4981 | 55° | 2.0347 | 1.2587 | 85° | 3.8317 | 1.0127 |
| 26° | 1.6557 | 1.4924 | 56° | 2.0571 | 1.2492 | 86° | 4.0528 | 1.0086 |
| 27° | 1.6627 | 1.4864 | 57° | 2.0804 | 1.2397 | 87° | 4.3387 | 1.0053 |
| 28° | 1.6701 | 1.4803 | 58° | 2.1047 | 1.2301 | 88° | 4.7427 | 1.0026 |
| 29° | 1.6777 | 1.4740 | 59° | 2.1300 | 1.2206 | 89° | 5.4349 | 1.0008 |

## Values of $F(k, \phi)$ for Certain Values of $k$ and $\phi$.

$$F(k, \phi) = \int_0^{\phi} \frac{dz}{\sqrt{1 - k^2 \sin^2 z}}.$$

| $\phi$ | $\alpha = \sin^{-1} k.$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 0° | 10° | 15° | 30° | 45° | 60° | 75° | 80° | 90° |
| 1° | 0.0174 | 0.0174 | 0.0174 | 0.0174 | 0.0174 | 0.0174 | 0.0174 | 0.0174 | 0.0174 |
| 2° | 0.0349 | 0.0349 | 0.0349 | 0.0349 | 0.0349 | 0.0349 | 0.0349 | 0.0349 | 0.0349 |
| 3° | 0.0524 | 0.0524 | 0.0524 | 0.0524 | 0.0524 | 0.0524 | 0.0524 | 0.0524 | 0.0524 |
| 4° | 0.0698 | 0.0698 | 0.0698 | 0.0698 | 0.0698 | 0.0699 | 0.0699 | 0.0699 | 0.0699 |
| 5° | 0.0873 | 0.0873 | 0.0873 | 0.0873 | 0.0873 | 0.0874 | 0.0874 | 0.0874 | 0.0874 |
| 10° | 0.1745 | 0.1746 | 0.1746 | 0.1748 | 0.1750 | 0.1752 | 0.1754 | 0.1754 | 0.1754 |
| 15° | 0.2618 | 0.2619 | 0.2620 | 0.2625 | 0.2633 | 0.2641 | 0.2646 | 0.2647 | 0.2648 |
| 20° | 0.3491 | 0.3493 | 0.3495 | 0.3508 | 0.3526 | 0.3545 | 0.3559 | 0.3562 | 0.3564 |
| 25° | 0.4363 | 0.4367 | 0.4372 | 0.4397 | 0.4433 | 0.4470 | 0.4498 | 0.4504 | 0.4509 |
| 30° | 0.5236 | 0.5243 | 0.5251 | 0.5294 | 0.5356 | 0.5422 | 0.5474 | 0.5484 | 0.5493 |
| 35° | 0.6109 | 0.6119 | 0.6132 | 0.6200 | 0.6300 | 0.6408 | 0.6495 | 0.6513 | 0.6528 |
| 40° | 0.6981 | 0.6997 | 0.7016 | 0.7116 | 0.7267 | 0.7436 | 0.7574 | 0.7604 | 0.7629 |
| 45° | 0.7854 | 0.7876 | 0.7902 | 0.8044 | 0.8260 | 0.8512 | 0.8727 | 0.8774 | 0.8814 |
| 50° | 0.8727 | 0.8756 | 0.8792 | 0.8982 | 0.9283 | 0.9646 | 0.9971 | 1.0044 | 1.0107 |
| 55° | 0.9599 | 0.9637 | 0.9683 | 0.9933 | 1.0337 | 1.0848 | 1.1331 | 1.1444 | 1.1542 |
| 60° | 1.0472 | 1.0519 | 1.0577 | 1.0896 | 1.1424 | 1.2125 | 1.2837 | 1.3014 | 1.3170 |
| 65° | 1.1345 | 1.1402 | 1.1474 | 1.1869 | 1.2545 | 1.3489 | 1.4532 | 1.4810 | 1.5064 |
| 70° | 1.2217 | 1.2286 | 1.2373 | 1.2853 | 1.3697 | 1.4944 | 1.6468 | 1.6918 | 1.7354 |
| 75° | 1.3090 | 1.3171 | 1.3273 | 1.3846 | 1.4879 | 1.6492 | 1.8714 | 1.9468 | 2.0276 |
| 80° | 1.3963 | 1.4056 | 1.4175 | 1.4846 | 1.6085 | 1.8125 | 2.1339 | 2.2653 | 2.4362 |
| 85° | 1.4835 | 1.4942 | 1.5078 | 1.5850 | 1.7308 | 1.9826 | 2.4366 | 2.6694 | 3.1313 |
| 86° | 1.5010 | 1.5120 | 1.5259 | 1.6052 | 1.7554 | 2.0172 | 2.5013 | 2.7612 | 3.3547 |
| 87° | 1.5184 | 1.5297 | 1.5439 | 1.6253 | 1.7801 | 2.0519 | 2.5670 | 2.8561 | 3.6425 |
| 88° | 1.5359 | 1.5474 | 1.5620 | 1.6454 | 1.8047 | 2.0867 | 2.6336 | 2.9537 | 4.0481 |
| 89° | 1.5533 | 1.5651 | 1.5801 | 1.6656 | 1.8294 | 2.1216 | 2.7007 | 3.0530 | 4.7414 |
| 90° | 1.5708 | 1.5828 | 1.5981 | 1.6858 | 1.8541 | 2.1565 | 2.7681 | 3.1534 | Inf. |

## Values of $E(k, \phi)$ for Certain Values of $k$ and $\phi$.

$$E(k, \phi) = \int_0^{\phi} \sqrt{1 - k^2 \sin^2 z} \cdot dz.$$

| $\phi$ | $\alpha = \sin^{-1} k.$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 0° | 10° | 15° | 30° | 45° | 60° | 75° | 80° | 90° |
| 1° | 0.0174 | 0.0174 | 0.0174 | 0.0174 | 0.0174 | 0.0174 | 0.0174 | 0.0174 | 0.0174 |
| 2° | 0.0349 | 0.0349 | 0.0349 | 0.0349 | 0.0349 | 0.0349 | 0.0349 | 0.0349 | 0.0349 |
| 3° | 0.0524 | 0.0524 | 0.0524 | 0.0524 | 0.0524 | 0.0523 | 0.0523 | 0.0523 | 0.0523 |
| 4° | 0.0698 | 0.0698 | 0.0698 | 0.0698 | 0.0698 | 0.0698 | 0.0698 | 0.0698 | 0.0698 |
| 5° | 0.0873 | 0.0873 | 0.0873 | 0.0872 | 0.0872 | 0.0872 | 0.0872 | 0.0872 | 0.0872 |
| 10° | 0.1745 | 0.1745 | 0.1745 | 0.1743 | 0.1741 | 0.1739 | 0.1737 | 0.1737 | 0.1736 |
| 15° | 0.2618 | 0.2617 | 0.2616 | 0.2611 | 0.2603 | 0.2596 | 0.2590 | 0.2589 | 0.2588 |
| 20° | 0.3491 | 0.3489 | 0.3486 | 0.3473 | 0.3456 | 0.3438 | 0.3425 | 0.3422 | 0.3420 |
| 25° | 0.4363 | 0.4359 | 0.4354 | 0.4330 | 0.4296 | 0.4261 | 0.4236 | 0.4230 | 0.4226 |
| 30° | 0.5236 | 0.5229 | 0.5221 | 0.5179 | 0.5120 | 0.5061 | 0.5016 | 0.5007 | 0.5000 |
| 35° | 0.6109 | 0.6098 | 0.6085 | 0.6019 | 0.5928 | 0.5833 | 0.5762 | 0.5748 | 0.5736 |
| 40° | 0.6981 | 0.6966 | 0.6947 | 0.6851 | 0.6715 | 0.6575 | 0.6468 | 0.6446 | 0.6428 |
| 45° | 0.7854 | 0.7832 | 0.7806 | 0.7672 | 0.7482 | 0.7282 | 0.7129 | 0.7097 | 0.7071 |
| 50° | 0.8727 | 0.8698 | 0.8663 | 0.8483 | 0.8226 | 0.7954 | 0.7741 | 0.7697 | 0.7660 |
| 55° | 0.9599 | 0.9562 | 0.9517 | 0.9284 | 0.8949 | 0.8588 | 0.8302 | 0.8242 | 0.8192 |
| 60° | 1.0472 | 1.0426 | 1.0368 | 1.0076 | 0.9650 | 0.9184 | 0.8808 | 0.8728 | 0.8660 |
| 65° | 1.1345 | 1.1288 | 1.1218 | 1.0858 | 1.0329 | 0.9743 | 0.9258 | 0.9152 | 0.9063 |
| 70° | 1.2217 | 1.2149 | 1.2065 | 1.1632 | 1.0990 | 1.0266 | 0.9652 | 0.9514 | 0.9397 |
| 75° | 1.3090 | 1.3010 | 1.2911 | 1.2399 | 1.1635 | 1.0759 | 0.9992 | 0.9814 | 0.9659 |
| 80° | 1.3963 | 1.3870 | 1.3755 | 1.3161 | 1.2266 | 1.1225 | 1.0282 | 1.0054 | 0.9848 |
| 85° | 1.4835 | 1.4729 | 1.4598 | 1.3919 | 1.2889 | 1.1673 | 1.0534 | 1.0244 | 0.9962 |
| 86° | 1.5010 | 1.4901 | 1.4767 | 1.4070 | 1.3012 | 1.1761 | 1.0581 | 1.0277 | 0.9976 |
| 87° | 1.5184 | 1.5073 | 1.4936 | 1.4221 | 1.3136 | 1.1848 | 1.0628 | 1.0309 | 0.9986 |
| 88° | 1.5359 | 1.5245 | 1.5104 | 1.4372 | 1.3260 | 1.1936 | 1.0674 | 1.0340 | 0.9994 |
| 89° | 1.5533 | 1.5417 | 1.5273 | 1.4524 | 1.3383 | 1.2023 | 1.0719 | 1.0371 | 0.9998 |
| 90° | 1.5708 | 1.5589 | 1.5442 | 1.4675 | 1.3506 | 1.2111 | 1.0764 | 1.0401 | 1.0000 |

## Hyperbolic Sines $[\sinh x = \frac{1}{2}(e^x - e^{-x})]$.

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Avg. diff. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | .0000 | .0100 | .0200 | .0300 | .0400 | .0500 | .0600 | .0701 | .0801 | .0901 | 100 |
| 1 | .1002 | .1102 | .1203 | .1304 | .1405 | .1506 | .1607 | .1708 | .1810 | .1911 | 101 |
| 2 | .2013 | .2115 | .2218 | .2320 | .2423 | .2526 | .2629 | .2733 | .2837 | .2941 | 103 |
| 3 | .3045 | .3150 | .3255 | .3360 | .3466 | .3572 | .3678 | .3785 | .3892 | .4000 | 106 |
| 4 | .4108 | .4216 | .4325 | .4434 | .4543 | .4653 | .4764 | .4875 | .4986 | .5098 | 110 |
| 0.5 | .5211 | .5324 | .5438 | .5552 | .5666 | .5782 | .5897 | .6014 | .6131 | .6248 | 116 |
| 6 | .6367 | .6485 | .6605 | .6725 | .6846 | .6967 | .7090 | .7213 | .7336 | .7461 | 122 |
| 7 | .7586 | .7712 | .7838 | .7966 | .8094 | .8223 | .8353 | .8484 | .8615 | .8748 | 130 |
| 8 | .8881 | .9015 | .9150 | .9286 | .9423 | .9561 | .9700 | .9840 | .9981 | 1.012 | 138 |
| 9 | 1.027 | 1.041 | 1.055 | 1.070 | 1.085 | 1.099 | 1.114 | 1.129 | 1.145 | 1.160 | 15 |
| 1.0 | 1.175 | 1.191 | 1.206 | 1.222 | 1.238 | 1.254 | 1.270 | 1.286 | 1.303 | 1.319 | 16 |
| 1 | 1.336 | 1.352 | 1.369 | 1.386 | 1.403 | 1.421 | 1.438 | 1.456 | 1.474 | 1.491 | 17 |
| 2 | 1.509 | 1.528 | 1.546 | 1.564 | 1.583 | 1.602 | 1.621 | 1.640 | 1.659 | 1.679 | 19 |
| 3 | 1.698 | 1.718 | 1.738 | 1.758 | 1.779 | 1.799 | 1.820 | 1.841 | 1.862 | 1.883 | 21 |
| 4 | 1.904 | 1.926 | 1.948 | 1.970 | 1.992 | 2.014 | 2.037 | 2.060 | 2.083 | 2.106 | 22 |
| 1.5 | 2.129 | 2.153 | 2.177 | 2.201 | 2.225 | 2.250 | 2.274 | 2.299 | 2.324 | 2.350 | 25 |
| 6 | 2.376 | 2.401 | 2.428 | 2.454 | 2.481 | 2.507 | 2.535 | 2.562 | 2.590 | 2.617 | 27 |
| 7 | 2.646 | 2.674 | 2.703 | 2.732 | 2.761 | 2.790 | 2.820 | 2.850 | 2.881 | 2.911 | 30 |
| 8 | 2.942 | 2.973 | 3.005 | 3.037 | 3.069 | 3.101 | 3.134 | 3.167 | 3.200 | 3.234 | 33 |
| 9 | 3.268 | 3.303 | 3.337 | 3.372 | 3.408 | 3.443 | 3.479 | 3.516 | 3.552 | 3.589 | 36 |
| 2.0 | 3.627 | 3.665 | 3.703 | 3.741 | 3.780 | 3.820 | 3.859 | 3.899 | 3.940 | 3.981 | 39 |
| 1 | 4.022 | 4.064 | 4.106 | 4.148 | 4.191 | 4.234 | 4.278 | 4.322 | 4.367 | 4.412 | 44 |
| 2 | 4.457 | 4.503 | 4.549 | 4.596 | 4.643 | 4.691 | 4.739 | 4.788 | 4.837 | 4.887 | 48 |
| 3 | 4.937 | 4.988 | 5.039 | 5.090 | 5.142 | 5.195 | 5.248 | 5.302 | 5.356 | 5.411 | 53 |
| 4 | 5.466 | 5.522 | 5.578 | 5.635 | 5.693 | 5.751 | 5.810 | 5.869 | 5.929 | 5.989 | 58 |
| 2.5 | 6.050 | 6.112 | 6.174 | 6.237 | 6.300 | 6.365 | 6.429 | 6.495 | 6.561 | 6.627 | 64 |
| 6 | 6.695 | 6.763 | 6.831 | 6.901 | 6.971 | 7.042 | 7.113 | 7.185 | 7.258 | 7.332 | 71 |
| 7 | 7.406 | 7.481 | 7.557 | 7.634 | 7.711 | 7.789 | 7.868 | 7.948 | 8.028 | 8.110 | 79 |
| 8 | 8.192 | 8.275 | 8.359 | 8.443 | 8.529 | 8.615 | 8.702 | 8.790 | 8.879 | 8.969 | 87 |
| 9 | 9.060 | 9.151 | 9.244 | 9.337 | 9.431 | 9.527 | 9.623 | 9.720 | 9.819 | 9.918 | 96 |
| 3.0 | 10.02 | 10.12 | 10.22 | 10.32 | 10.43 | 10.53 | 10.64 | 10.75 | 10.86 | 10.97 | 11 |
| 1 | 11.08 | 11.19 | 11.30 | 11.42 | 11.53 | 11.65 | 11.76 | 11.88 | 12.00 | 12.12 | 12 |
| 2 | 12.25 | 12.37 | 12.49 | 12.62 | 12.75 | 12.88 | 13.01 | 13.14 | 13.27 | 13.40 | 13 |
| 3 | 13.54 | 13.67 | 13.81 | 13.95 | 14.09 | 14.23 | 14.38 | 14.52 | 14.67 | 14.82 | 14 |
| 4 | 14.97 | 15.12 | 15.27 | 15.42 | 15.58 | 15.73 | 15.89 | 16.05 | 16.21 | 16.38 | 16 |
| 3.5 | 16.54 | 16.71 | 16.88 | 17.05 | 17.22 | 17.39 | 17.57 | 17.74 | 17.92 | 18.10 | 17 |
| 6 | 18.29 | 18.47 | 18.66 | 18.84 | 19.03 | 19.22 | 19.42 | 19.62 | 19.81 | 20.01 | 19 |
| 7 | 20.21 | 20.41 | 20.62 | 20.83 | 21.04 | 21.25 | 21.46 | 21.68 | 21.90 | 22.12 | 21 |
| 8 | 22.34 | 22.56 | 22.79 | 23.02 | 23.25 | 23.49 | 23.72 | 23.96 | 24.20 | 24.45 | 24 |
| 9 | 24.69 | 24.94 | 25.19 | 25.44 | 25.70 | 25.96 | 26.22 | 26.48 | 26.75 | 27.02 | 26 |
| 4.0 | 27.29 | 27.56 | 27.84 | 28.12 | 28.40 | 28.69 | 28.98 | 29.27 | 29.56 | 29.86 | 29 |
| 1 | 30.16 | 30.47 | 30.77 | 31.08 | 31.39 | 31.71 | 32.03 | 32.35 | 32.68 | 33.00 | 32 |
| 2 | 33.34 | 33.67 | 34.01 | 34.35 | 34.70 | 35.05 | 35.40 | 35.75 | 36.11 | 36.48 | 35 |
| 3 | 36.84 | 37.21 | 37.59 | 37.97 | 38.35 | 38.73 | 39.12 | 39.52 | 39.91 | 40.31 | 39 |
| 4 | 40.72 | 41.13 | 41.54 | 41.96 | 42.38 | 42.81 | 43.24 | 43.67 | 44.11 | 44.56 | 43 |
| 4.5 | 45.00 | 45.46 | 45.91 | 46.37 | 46.84 | 47.31 | 47.79 | 48.27 | 48.75 | 49.24 | 47 |
| 6 | 49.74 | 50 24 | 50.74 | 51.25 | 51.77 | 52.29 | 52.81 | 53.34 | 53.88 | 54.42 | 52 |
| 7 | 54.97 | 55.52 | 56.08 | 56.64 | 57.21 | 57.79 | 58.37 | 58.96 | 59.55 | 60.15 | 58 |
| 8 | 60.75 | 61.36 | 61.98 | 62.60 | 63.23 | 63.87 | 64.51 | 65.16 | 65.81 | 66.47 | 64 |
| 9 | 67.14 | 67.82 | 68.50 | 69.19 | 69.88 | 70.58 | 71.29 | 72.01 | 72.73 | 73.46 | 71 |
| 5.0 | 74.20 | | | | | | | | | | |

If $x > 5$, $\sinh x = \frac{1}{2}(e^x)$ and $\log_{10} \sinh x = (0.4343)x + 0.6990 - 1$, correct to four significant figures.

## Hyperbolic Cosines $[\cosh x = \frac{1}{2}(e^x + e^{-x})]$.

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Avg. diff. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 1.000 | 1.000 | 1.000 | 1.000 | 1.001 | 1.001 | 1.002 | 1.002 | 1.003 | 1.004 | 1 |
| 1 | 1.005 | 1.006 | 1.007 | 1.008 | 1.010 | 1.011 | 1.013 | 1.014 | 1.016 | 1.018 | 2 |
| 2 | 1.020 | 1.022 | 1.024 | 1.027 | 1.029 | 1.031 | 1.034 | 1.037 | 1.039 | 1.042 | 3 |
| 3 | 1.045 | 1.048 | 1.052 | 1.055 | 1.058 | 1.062 | 1.066 | 1.069 | 1.073 | 1.077 | 4 |
| 4 | 1.081 | 1.085 | 1.090 | 1.094 | 1.098 | 1.103 | 1.108 | 1.112 | 1.117 | 1.122 | 5 |
| 0.5 | 1.128 | 1.133 | 1.138 | 1.144 | 1.149 | 1.155 | 1.161 | 1.167 | 1.173 | 1.179 | 6 |
| 6 | 1.185 | 1.192 | 1.198 | 1.205 | 1.212 | 1.219 | 1.226 | 1.233 | 1.240 | 1.248 | 7 |
| 7 | 1.255 | 1.263 | 1.271 | 1.278 | 1.287 | 1.295 | 1.303 | 1.311 | 1.320 | 1.329 | 8 |
| 8 | 1.337 | 1.346 | 1.355 | 1.365 | 1.374 | 1.384 | 1.393 | 1.403 | 1.413 | 1.423 | 10 |
| 9 | 1.433 | 1.443 | 1.454 | 1.465 | 1.475 | 1.486 | 1.497 | 1.509 | 1.520 | 1.531 | 11 |
| 1.0 | 1.543 | 1.555 | 1.567 | 1.579 | 1.591 | 1.604 | 1.616 | 1.629 | 1.642 | 1.655 | 13 |
| 1 | 1.669 | 1.682 | 1.696 | 1.709 | 1.723 | 1.737 | 1.752 | 1.766 | 1.781 | 1.796 | 14 |
| 2 | 1.811 | 1.826 | 1.841 | 1.857 | 1.872 | 1.888 | 1.905 | 1.921 | 1.937 | 1.954 | 16 |
| 3 | 1.971 | 1.988 | 2.005 | 2.023 | 2.040 | 2.058 | 2.076 | 2.095 | 2.113 | 2.132 | 18 |
| 4 | 2.151 | 2.170 | 2.189 | 2.209 | 2.229 | 2.249 | 2.269 | 2.290 | 2.310 | 2.331 | 20 |
| 1.5 | 2.352 | 2.374 | 2.395 | 2.417 | 2.439 | 2.462 | 2.484 | 2.507 | 2.530 | 2.554 | 23 |
| 6 | 2.577 | 2.601 | 2.625 | 2.650 | 2.675 | 2.700 | 2.725 | 2.750 | 2.776 | 2.802 | 25 |
| 7 | 2.828 | 2.855 | 2.882 | 2.909 | 2.936 | 2.964 | 2.992 | 3.021 | 3.049 | 3.078 | 28 |
| 8 | 3.107 | 3.137 | 3.167 | 3.197 | 3.228 | 3.259 | 3.290 | 3.321 | 3.353 | 3.385 | 31 |
| 9 | 3.418 | 3.451 | 3.484 | 3.517 | 3.551 | 3.585 | 3.620 | 3.655 | 3.690 | 3.726 | 34 |
| 2.0 | 3.762 | 3.799 | 3.835 | 3.873 | 3.910 | 3.948 | 3.987 | 4.026 | 4.065 | 4.104 | 38 |
| 1 | 4.144 | 4.185 | 4.226 | 4.267 | 4.309 | 4.351 | 4.393 | 4.436 | 4.480 | 4.524 | 42 |
| 2 | 4.568 | 4.613 | 4.658 | 4.704 | 4.750 | 4.797 | 4.844 | 4.891 | 4.939 | 4.988 | 47 |
| 3 | 5.037 | 5.087 | 5.137 | 5.188 | 5.239 | 5.290 | 5.343 | 5.395 | 5.449 | 5.503 | 52 |
| 4 | 5.557 | 5.612 | 5.667 | 5.723 | 5.780 | 5.837 | 5.895 | 5.954 | 6.013 | 6.072 | 58 |
| 2.5 | 6.132 | 6.193 | 6.255 | 6.317 | 6.379 | 6.443 | 6.507 | 6.571 | 6.636 | 6.702 | 64 |
| 6 | 6.769 | 6.836 | 6.904 | 6.973 | 7.042 | 7.112 | 7.183 | 7.255 | 7.327 | 7.400 | 70 |
| 7 | 7.473 | 7.548 | 7.623 | 7.699 | 7.776 | 7.853 | 7.932 | 8.011 | 8.091 | 8.171 | 78 |
| 8 | 8.253 | 8.335 | 8.418 | 8.502 | 8.587 | 8.673 | 8.759 | 8.847 | 8.935 | 9.024 | 86 |
| 9 | 9.115 | 9.206 | 9.298 | 9.391 | 9.484 | 9.579 | 9.675 | 9.772 | 9.869 | 9.968 | 95 |
| 3.0 | 10.07 | 10.17 | 10.27 | 10.37 | 10.48 | 10.58 | 10.69 | 10.79 | 10.90 | 11.01 | 11 |
| 1 | 11.12 | 11.23 | 11.35 | 11.46 | 11.57 | 11.69 | 11.81 | 11.92 | 12.04 | 12.16 | 12 |
| 2 | 12.29 | 12.41 | 12.53 | 12.66 | 12.79 | 12.91 | 13.04 | 13.17 | 13.31 | 13.44 | 13 |
| 3 | 13.57 | 13.71 | 13.85 | 13.99 | 14.13 | 14.27 | 14.41 | 14.56 | 14.70 | 14.85 | 14 |
| 4 | 15.00 | 15.15 | 15.30 | 15.45 | 15.61 | 15.77 | 15.92 | 16.08 | 16.25 | 16.41 | 16 |
| 3.5 | 16.57 | 16.74 | 16.91 | 17.08 | 17.25 | 17.42 | 17.60 | 17.77 | 17.95 | 18.13 | 17 |
| 6 | 18.31 | 18.50 | 18.68 | 18.87 | 19.06 | 19.25 | 19.44 | 19.64 | 19.84 | 20.03 | 19 |
| 7 | 20.24 | 20.44 | 20.64 | 20.85 | 21.06 | 21.27 | 21.49 | 21.70 | 21.92 | 22.14 | 21 |
| 8 | 22.36 | 22.59 | 22.81 | 23.04 | 23.27 | 23.51 | 23.74 | 23.98 | 24.22 | 24.47 | 23 |
| 9 | 24.71 | 24.96 | 25.21 | 25.46 | 25.72 | 25.98 | 26.24 | 26.50 | 26.77 | 27.04 | 26 |
| 4.0 | 27.31 | 27.58 | 27.86 | 28.14 | 28.42 | 28.71 | 29.00 | 29.29 | 29.58 | 29.88 | 29 |
| 1 | 30.18 | 30.48 | 30.79 | 31.10 | 31.41 | 31.72 | 32.04 | 32.37 | 32.69 | 33.02 | 32 |
| 2 | 33.35 | 33.69 | 34.02 | 34.37 | 34.71 | 35.06 | 35.41 | 35.77 | 36.13 | 36.49 | 35 |
| 3 | 36.86 | 37.23 | 37.60 | 37.98 | 38.36 | 38.75 | 39.13 | 39.53 | 39.93 | 40.33 | 39 |
| 4 | 40.73 | 41.14 | 41.55 | 41.97 | 42.39 | 42.82 | 43.25 | 43.68 | 44.12 | 44.57 | 43 |
| 4.5 | 45.01 | 45.47 | 45.92 | 46.38 | 46.85 | 47.32 | 47.80 | 48.28 | 48.76 | 49.25 | 47 |
| 6 | 49.75 | 50.25 | 50.75 | 51.26 | 51.78 | 52.30 | 52.82 | 53.35 | 53.89 | 54.43 | 52 |
| 7 | 54.98 | 55.53 | 56.09 | 56.65 | 57.22 | 57.80 | 58.38 | 58.96 | 59.56 | 60 15 | 58 |
| 8 | 60.76 | 61.37 | 61.99 | 62.61 | 63.24 | 63.87 | 64.52 | 65.16 | 65.82 | 66.48 | 64 |
| 9 | 67.15 | 67.82 | 68.50 | 69.19 | 69.89 | 70.59 | 71.30 | 72.02 | 72.74 | 73.47 | 71 |
| 5.0 | 74.21 | | | | | | | | | | |

If $x > 5$, $\cosh x = \frac{1}{2}(e^x)$ and $\log_{10} \cosh x = (0.4343)x + 0.6990 - 1$, correct to four significant figures.

Hyperbolic Tangents [$\tanh x = (e^x - e^{-x})/(e^x + e^{-x}) = \sinh x/\cosh x$].

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Avg. diff. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | .0000 | .0100 | .0200 | .0300 | .0400 | .0500 | .0599 | .0699 | .0798 | .0898 | 100 |
| 1 | .0997 | .1096 | .1194 | .1293 | .1391 | .1489 | .1587 | .1684 | .1781 | .1878 | 98 |
| 2 | .1974 | .2070 | .2165 | .2260 | .2355 | .2449 | .2543 | .2636 | .2729 | .2821 | 94 |
| 3 | .2913 | .3004 | .3095 | .3185 | .3275 | .3364 | .3452 | .3540 | .3627 | .3714 | 89 |
| 4 | .3800 | .3885 | .3969 | .4053 | .4136 | .4219 | .4301 | .4382 | .4462 | .4542 | 82 |
| 0.5 | .4621 | .4700 | .4777 | .4854 | .4930 | .5005 | .5080 | .5154 | .5227 | .5299 | 75 |
| 6 | .5370 | .5441 | .5511 | .5581 | .5649 | .5717 | .5784 | .5850 | .5915 | .5980 | 67 |
| 7 | .6044 | .6107 | .6169 | .6231 | .6291 | .6352 | .6411 | .6469 | .6527 | .6584 | 60 |
| 8 | .6640 | .6696 | .6751 | .6805 | .6858 | .6911 | .6963 | .7014 | .7064 | .7114 | 52 |
| 9 | .7163 | .7211 | .7259 | .7306 | .7352 | .7398 | .7443 | .7487 | .7531 | .7574 | 45 |
| 1.0 | .7616 | .7658 | .7699 | .7739 | .7779 | .7818 | .7857 | .7895 | .7932 | .7969 | 39 |
| 1 | .8005 | .8041 | .8076 | .8110 | .8144 | .8178 | .8210 | .8243 | .8275 | .8306 | 33 |
| 2 | .8337 | .8367 | .8397 | .8426 | .8455 | .8483 | .8511 | .8538 | .8565 | .8591 | 28 |
| 3 | .8617 | .8643 | .8668 | .8693 | .8717 | .8741 | .8764 | .8787 | .8810 | .8832 | 24 |
| 4 | .8854 | .8875 | .8896 | .8917 | .8937 | .8957 | .8977 | .8996 | .9015 | .9033 | 20 |
| 1.5 | .9052 | .9069 | .9087 | .9104 | .9121 | .9138 | .9154 | .9170 | .9186 | .9202 | 17 |
| 6 | .9217 | .9232 | .9246 | .9261 | .9275 | .9289 | .9302 | .9316 | .9329 | .9342 | 14 |
| 7 | .9354 | .9367 | .9379 | .9391 | .9402 | .9414 | .9425 | .9436 | .9447 | .9458 | 11 |
| 8 | .9468 | .9478 | .9488 | .9498 | .9508 | .9518 | .9527 | .9536 | .9545 | .9554 | 9 |
| 9 | .9562 | .9571 | .9579 | .9587 | .9595 | .9603 | .9611 | .9619 | .9626 | .9633 | 8 |
| 2.0 | .9640 | .9647 | .9654 | .9661 | .9668 | .9674 | .9680 | .9687 | .9693 | .9699 | 6 |
| 1 | .9705 | .9710 | .9716 | .9722 | .9727 | .9732 | .9738 | .9743 | .9748 | .9753 | 5 |
| 2 | .9757 | .9762 | .9767 | .9771 | .9776 | .9780 | .9785 | .9789 | .9793 | .9797 | 4 |
| 3 | .9801 | .9805 | .9809 | .9812 | .9816 | .9820 | .9823 | .9827 | .9830 | .9834 | 4 |
| 4 | .9837 | .9840 | .9843 | .9846 | .9849 | .9852 | .9855 | .9858 | .9861 | .9863 | 3 |
| 2.5 | .9866 | .9869 | .9871 | .9874 | .9876 | .9879 | .9881 | .9884 | .9886 | .9888 | 2 |
| 6 | .9890 | .9892 | .9895 | .9897 | .9899 | .9901 | .9903 | .9905 | .9906 | .9908 | 2 |
| 7 | .9910 | .9912 | .9914 | .9915 | .9917 | .9919 | .9920 | .9922 | .9923 | .9925 | 2 |
| 8 | .9926 | .9928 | .9929 | .9931 | .9932 | .9933 | .9935 | .9936 | .9937 | .9938 | 1 |
| 2.9 | .9940 | .9941 | .9942 | .9943 | .9944 | .9945 | .9946 | .9947 | .9949 | .9950 | 1 |
| 3. | .9951 | .9959 | .9967 | .9973 | .9978 | .9982 | .9985 | .9988 | .9990 | .9992 | 4 |
| 4. | .9993 | .9995 | .9996 | .9996 | .9997 | .9998 | .9998 | .9998 | .9999 | .9999 | 1 |
| 5. | .9999 | If $x > 5$, $\tanh x = 1.0000$ to four decimal places. | | | | | | | | | |

Multiples of 0.4343 ($0.43429448 = \log_{10} e$).

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0. | 0.0000 | 0.0434 | 0.0869 | 0.1303 | 0.1737 | 0.2171 | 0.2606 | 0.3040 | 0.3474 | 0.3909 |
| 1. | 0.4343 | 0.4777 | 0.5212 | 0.5646 | 0.6080 | 0.6514 | 0.6949 | 0.7383 | 0.7817 | 0.8252 |
| 2. | 0.8686 | 0.9120 | 0.9554 | 0.9989 | 1.0423 | 1.0857 | 1.1292 | 1.1726 | 1.2160 | 1.2595 |
| 3. | 1.3029 | 1.3463 | 1.3897 | 1.4332 | 1.4766 | 1.5200 | 1.5635 | 1.6069 | 1.6503 | 1.6937 |
| 4. | 1.7372 | 1.7806 | 1.8240 | 1.8675 | 1.9109 | 1.9543 | 1.9978 | 2.0412 | 2.0846 | 2.1280 |
| 5. | 2.1715 | 2.2149 | 2.2583 | 2.3018 | 2.3452 | 2.3886 | 2.4320 | 2.4755 | 2.5189 | 2.5623 |
| 6. | 2.6058 | 2.6492 | 2.6926 | 2.7361 | 2.7795 | 2.8229 | 2.8663 | 2.9098 | 2.9532 | 2.9966 |
| 7. | 3.0401 | 3.0835 | 3.1269 | 3.1703 | 3.2138 | 3.2572 | 3.3006 | 3.3441 | 3.3875 | 3.4309 |
| 8. | 3.4744 | 3.5178 | 3.5612 | 3.6046 | 3.6481 | 3.6915 | 3.7349 | 3.7784 | 3.8218 | 3.8652 |
| 9. | 3.9087 | 3.9521 | 3.9955 | 4.0389 | 4.0824 | 4.1258 | 4.1692 | 4.2127 | 4.2561 | 4.2995 |

Multiples of 2.3026 ($2.3025851 = 1/0.4343$).

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0. | 0.0000 | 0.2303 | 0.4605 | 0.6908 | 0.9210 | 1.1513 | 1.3816 | 1.6118 | 1.8421 | 2.0723 |
| 1. | 2.3026 | 2.5328 | 2.7631 | 2.9934 | 3.2236 | 3.4539 | 3.6841 | 3.9144 | 4.1447 | 4.3749 |
| 2. | 4.6052 | 4.8354 | 5.0657 | 5.2959 | 5.5262 | 5.7565 | 5.9867 | 6.2170 | 6.4472 | 6.6775 |
| 3. | 6.9078 | 7.1380 | 7.3683 | 7.5985 | 7.8288 | 8.0590 | 8.2893 | 8.5196 | 8.7498 | 8.9801 |
| 4. | 9.2103 | 9.4406 | 9.6709 | 9.9011 | 10.131 | 10.362 | 10.592 | 10.822 | 11.052 | 11.283 |
| 5. | 11.513 | 11.743 | 11.973 | 12.204 | 12.434 | 12.664 | 12.894 | 13.125 | 13.355 | 13.585 |
| 6. | 13.816 | 14.046 | 14.276 | 14.506 | 14.737 | 14.967 | 15.197 | 15.427 | 15.658 | 15.888 |
| 7. | 16.118 | 16.348 | 16.579 | 16.809 | 17.039 | 17.269 | 17.500 | 17.730 | 17.960 | 18.190 |
| 8. | 18.421 | 18.651 | 18.881 | 19.111 | 19.342 | 19.572 | 19.802 | 20.032 | 20.263 | 20.493 |
| 9. | 20.723 | 20.954 | 21.184 | 21.414 | 21.644 | 21.875 | 22.105 | 22.335 | 22.565 | 22.796 |

## Exponentials [$e^n$ and $e^{-n}$].

| $n$ | $e^n$ | Diff. | $n$ | $e^n$ | Diff. | $n$ | $e^n$ | $n$ | $e^{-n}$ | Diff. | $n$ | $e^{-n}$ | $n$ | $e^{-n}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **0.00** | 1.000 | 10 | **0.50** | 1.649 | 16 | **1.0** | 2.718* | **0.00** | 1.000 | — 10 | **0.50** | .607 | **1.0** | .368* |
| .01 | 1.010 | 10 | .51 | 1.665 | 17 | .1 | 3.004 | .01 | 0.990 | — 10 | .51 | .600 | .1 | .333 |
| .02 | 1.020 | 10 | .52 | 1.682 | 17 | .2 | 3.320 | .02 | .980 | — 10 | .52 | .595 | .2 | .301 |
| .03 | 1.030 | 11 | .53 | 1.699 | 17 | .3 | 3.669 | .03 | .970 | — 9 | .53 | .589 | .3 | .273 |
| .04 | 1.041 | 10 | .54 | 1.716 | 17 | .4 | 4.055 | .04 | .961 | — 10 | .54 | .583 | .4 | .247 |
| 0.05 | 1.051 | 11 | 0.55 | 1.733 | 18 | 1.5 | 4.482 | 0.05 | .951 | — 9 | 0.55 | .577 | 1.5 | .223 |
| .06 | 1.062 | 11 | .56 | 1.751 | 17 | .6 | 4.953 | .06 | .942 | — 10 | .56 | .571 | .6 | .202 |
| .07 | 1.073 | 10 | .57 | 1.768 | 18 | .7 | 5.474 | .07 | .932 | — 9 | .57 | .566 | .7 | .183 |
| .08 | 1.083 | 11 | .58 | 1.786 | 18 | .8 | 6.050 | .08 | .923 | — 9 | .58 | .560 | .8 | .165 |
| .09 | 1.094 | 11 | .59 | 1.804 | 18 | .9 | 6.686 | .09 | .914 | — 9 | .59 | .554 | .9 | .150 |
| **0.10** | 1.105 | 11 | **0.60** | 1.822 | 18 | **2.0** | 7.389 | **0.10** | .905 | — 9 | **0.60** | .549 | **2.0** | .135 |
| .11 | 1.116 | 11 | .61 | 1.840 | 19 | .1 | 8.166 | .11 | .896 | — 9 | .61 | .543 | .1 | .122 |
| .12 | 1.127 | 12 | .62 | 1.859 | 19 | .2 | 9.025 | .12 | .887 | — 9 | .62 | .538 | .2 | .111 |
| .13 | 1.139 | 11 | .63 | 1.878 | 18 | .3 | 9.974 | .13 | .878 | — 9 | .63 | .533 | .3 | .100 |
| .14 | 1.150 | 12 | .64 | 1.896 | 20 | .4 | 11.02 | .14 | .869 | — 8 | .64 | .527 | .4 | .0907 |
| 0.15 | 1.162 | 12 | 0.65 | 1.916 | 19 | 2.5 | 12.18 | 0.15 | .861 | — 9 | 0.65 | .522 | 2.5 | .0821 |
| .16 | 1.174 | 11 | .66 | 1.935 | 19 | .6 | 13.46 | .16 | .852 | — 8 | .66 | .517 | .6 | .0743 |
| .17 | 1.185 | 12 | .67 | 1.954 | 20 | .7 | 14.88 | .17 | .844 | — 9 | .67 | .512 | .7 | .0672 |
| .18 | 1.197 | 12 | .68 | 1.974 | 20 | .8 | 16.44 | .18 | .835 | — 8 | .68 | .507 | .8 | .0608 |
| .19 | 1.209 | 12 | .69 | 1.994 | 20 | .9 | 18.17 | .19 | .827 | — 8 | .69 | .502 | .9 | .0550 |
| **0.20** | 1.221 | 13 | **0.70** | 2.014 | 20 | **3.0** | 20.09 | **0.20** | .819 | — 8 | **0.70** | .497 | **3.0** | .0498 |
| .21 | 1.234 | 12 | .71 | 2.034 | 20 | .1 | 22.20 | .21 | .811 | — 8 | .71 | .492 | .1 | .0450 |
| .22 | 1.246 | 13 | .72 | 2.054 | 21 | .2 | 24.53 | .22 | .803 | — 8 | .72 | .487 | .2 | .0408 |
| .23 | 1.259 | 12 | .73 | 2.075 | 21 | .3 | 27.11 | .23 | .795 | — 8 | .73 | .482 | .3 | .0369 |
| .24 | 1.271 | 13 | .74 | 2.096 | 21 | .4 | 29.96 | .24 | .787 | — 8 | .74 | .477 | .4 | .0334 |
| 0.25 | 1.284 | 13 | 0.75 | 2.117 | 21 | 3.5 | 33.12 | 0.25 | .779 | — 8 | 0.75 | .472 | 3.5 | .0302 |
| .26 | 1.297 | 13 | .76 | 2.138 | 22 | .6 | 36.60 | .26 | .771 | — 8 | .76 | .468 | .6 | .0273 |
| .27 | 1.310 | 13 | .77 | 2.160 | 21 | .7 | 40.45 | .27 | .763 | — 7 | .77 | .463 | .7 | .0247 |
| .28 | 1.323 | 13 | .78 | 2.181 | 22 | .8 | 44.70 | .28 | .756 | — 8 | .78 | .458 | .8 | .0224 |
| .29 | 1.336 | 14 | .79 | 2.203 | 23 | .9 | 49.40 | .29 | .748 | — 7 | .79 | .454 | .9 | .0202 |
| **0.30** | 1.350 | 13 | **0.80** | 2.226 | 22 | **4.0** | 54.60 | **0.30** | .741 | — 8 | **0.80** | .449 | **4.0** | .0183 |
| .31 | 1.363 | 14 | .81 | 2.248 | 22 | .1 | 60.34 | .31 | .733 | — 7 | .81 | .445 | .1 | .0166 |
| .32 | 1.377 | 14 | .82 | 2.270 | 23 | .2 | 66.69 | .32 | .726 | — 7 | .82 | .440 | .2 | .0150 |
| .33 | 1.391 | 14 | .83 | 2.293 | 23 | .3 | 73.70 | .33 | .719 | — 7 | .83 | .436 | .3 | .0136 |
| .34 | 1.405 | 14 | .84 | 2.316 | 24 | .4 | 81.45 | .34 | .712 | — 7 | .84 | .432 | .4 | .0123 |
| 0.35 | 1.419 | 14 | 0.85 | 2.340 | 23 | 4.5 | 90.02 | 0.35 | .705 | — 7 | 0.85 | .427 | 4.5 | .0111 |
| .36 | 1.433 | 15 | .86 | 2.363 | 24 | | | .36 | .698 | — 7 | .86 | .423 | | |
| .37 | 1.448 | 14 | .87 | 2.387 | 24 | 5.0 | 148.4 | .37 | .691 | — 7 | .87 | .419 | 5.0 | .00674 |
| .38 | 1.462 | 15 | .88 | 2.411 | 24 | 6.0 | 403.4 | .38 | .684 | — 7 | .88 | .415 | 6.0 | .00248 |
| .39 | 1.477 | 15 | .89 | 2.435 | 25 | 7.0 | 1097. | .39 | .677 | — 7 | .89 | .411 | 7.0 | .000912 |
| **0.40** | 1.492 | 15 | **0.90** | 2.460 | 24 | 8.0 | 2981. | **0.40** | .670 | — 6 | **0.90** | .407 | 8.0 | .000335 |
| .41 | 1.507 | 15 | .91 | 2.484 | 25 | 9.0 | 8103. | .41 | .664 | — 7 | .91 | .403 | 9.0 | .000123 |
| .42 | 1.522 | 15 | .92 | 2.509 | 26 | **10.0** | 22026. | .42 | .657 | — 6 | .92 | .399 | **10.0** | .000045 |
| .43 | 1.537 | 16 | .93 | 2.535 | 25 | $\pi/2$ | 4.810 | .43 | .651 | — 7 | .93 | .395 | $\pi/2$ | .208 |
| .44 | 1.553 | 15 | .94 | 2.560 | 26 | $2\pi/2$ | 23.14 | .44 | .644 | — 6 | .94 | .391 | $2\pi/2$ | .0432 |
| 0.45 | 1.568 | 16 | 0.95 | 2.586 | 26 | $3\pi/2$ | 111.3 | 0.45 | .638 | — 7 | 0.95 | .387 | $3\pi/2$ | .00898 |
| .46 | 1.584 | 16 | .96 | 2.612 | 26 | $4\pi/2$ | 535.5 | .46 | .631 | — 6 | .96 | .383 | $4\pi/2$ | .00187 |
| .47 | 1.600 | 16 | .97 | 2.638 | 26 | $5\pi/2$ | 2576. | .47 | .625 | — 6 | .97 | .379 | $5\pi/2$ | .000388 |
| .48 | 1.616 | 16 | .98 | 2.664 | 27 | $6\pi/2$ | 12392. | .48 | .619 | — 6 | .98 | .375 | $6\pi/2$ | .000081 |
| .49 | 1.632 | 17 | .99 | 2.691 | 27 | $7\pi/2$ | 59610. | .49 | .613 | — 6 | .99 | .372 | $7\pi/2$ | .000017 |
| **0.50** | 1.649 | | **1.00** | 2.718 | | $8\pi/2$ | 286751. | **0.50** | 0.607 | | **1.00** | .368 | $8\pi/2$ | .000003 |

* NOTE 1. — Do not interpolate in this column. $e = 2.71828$ $1/e = 0.367879$
$\log_{10} e = 0.4343$ $1/(0.4343) = 2.3026$ $\log_{10}(0.4343) = 1.6378$ $\log_{10}(e^n) = n(0.4343)$

NOTE 2. — This page and the three that precede it are taken from E. V. Huntington's Handbook of Mathematics for Engineers, published by the McGraw-Hill Book Company, Inc.

## The Common Logarithms of $e^x$ and $e^{-x}$.

| $x$ | $\log_{10} e^x$ | $\log_{10} e^{-x}$ |
|---|---|---|
| 0.00001 | 0.0000043429 | $\bar{1}$.9999956571 |
| 0.00002 | 0.0000086859 | $\bar{1}$.9999913141 |
| 0.00003 | 0.0000130288 | $\bar{1}$.9999869712 |
| 0.00004 | 0.0000173718 | $\bar{1}$.9999826282 |
| 0.00005 | 0.0000217147 | $\bar{1}$.9999782853 |
| 0.00006 | 0.0000260577 | $\bar{1}$.9999739423 |
| 0.00007 | 0.0000304006 | $\bar{1}$.9999695994 |
| 0.00008 | 0.0000347436 | $\bar{1}$.9999652564 |
| 0.00009 | 0.0000390865 | $\bar{1}$.9999609135 |
| 0.00010 | 0.0000434294 | $\bar{1}$.9999565706 |
| 0.00020 | 0.0000868589 | $\bar{1}$.9999131411 |
| 0.00030 | 0.0001302883 | $\bar{1}$.9998697117 |
| 0.00040 | 0.0001737178 | $\bar{1}$.9998262822 |
| 0.00050 | 0.0002171472 | $\bar{1}$.9997828528 |
| 0.00060 | 0.0002605767 | $\bar{1}$.9997394233 |
| 0.00070 | 0.0003040061 | $\bar{1}$.9996959939 |
| 0.00080 | 0.0003474356 | $\bar{1}$.9996525644 |
| 0.00090 | 0.0003908650 | $\bar{1}$.9996091350 |
| 0.00100 | 0.0004342945 | $\bar{1}$.9995657055 |
| 0.00200 | 0.0008685890 | $\bar{1}$.9991314110 |
| 0.00300 | 0.0013028834 | $\bar{1}$.9986971166 |
| 0.00400 | 0.0017371779 | $\bar{1}$.9982628221 |
| 0.00500 | 0.0021714724 | $\bar{1}$.9978285276 |
| 0.00600 | 0.0026057669 | $\bar{1}$.9973942331 |
| 0.00700 | 0.0030400614 | $\bar{1}$.9969599386 |
| 0.00800 | 0.0034743559 | $\bar{1}$.9965256441 |
| 0.00900 | 0.0039086503 | $\bar{1}$.9960913497 |
| 0.01000 | 0.0043429448 | $\bar{1}$.9956570552 |
| 0.02000 | 0.0086858896 | $\bar{1}$.9913141104 |
| 0.03000 | 0.0130288345 | $\bar{1}$.9869711655 |
| 0.04000 | 0.0173717793 | $\bar{1}$.9826282207 |
| 0.05000 | 0.0217147241 | $\bar{1}$.9782852759 |
| 0.06000 | 0.0260576689 | $\bar{1}$.9739423311 |
| 0.07000 | 0.0304006137 | $\bar{1}$.9695993863 |

| $x$ | $\log_{10} e^{x}$ | $\log_{10} e^{-x}$ |
|---|---|---|
| 0.08000 | 0.0347435586 | $\bar{1}$.9652564414 |
| 0.09000 | 0.0390865034 | $\bar{1}$.9609134966 |
| 0.10000 | 0.0434294482 | $\bar{1}$.9565705518 |
| 0.20000 | 0.0868588964 | $\bar{1}$.9131411036 |
| 0.30000 | 0.1302883446 | $\bar{1}$.8697116554 |
| 0.40000 | 0.1737177928 | $\bar{1}$.8262822072 |
| 0.50000 | 0.2171472410 | $\bar{1}$.7828527590 |
| 0.60000 | 0.2605766891 | $\bar{1}$.7394233109 |
| 0.70000 | 0.3040061373 | $\bar{1}$.6959938627 |
| 0.80000 | 0.3474355855 | $\bar{1}$.6525644145 |
| 0.90000 | 0.3908650337 | $\bar{1}$.6091349663 |
| 1.00000 | 0.4342944819 | $\bar{1}$.5657055181 |
| 2.00000 | 0.8685889638 | $\bar{1}$.1314110362 |
| 3.00000 | 1.3028834457 | $\bar{2}$.6971165543 |
| 4.00000 | 1.7371779276 | $\bar{2}$.2628220724 |
| 5.00000 | 2.1714724095 | $\bar{3}$.8285275905 |
| 6.00000 | 2.6057668914 | $\bar{3}$.3942331086 |
| 7.00000 | 3.0400613733 | $\bar{4}$.9599386267 |
| 8.00000 | 3.4743558552 | $\bar{4}$.5256441448 |
| 9.00000 | 3.9086503371 | $\bar{4}$.0913496629 |
| 10.00000 | 4.3429448190 | $\bar{5}$.6570551810 |
| 20.00000 | 8.6858896381 | $\bar{9}$.3141103619 |
| 30.00000 | 13.0288344571 | $\bar{14}$.9711655429 |
| 40.00000 | 17.3717792761 | $\bar{18}$.6282207239 |
| 50.00000 | 21.7147240952 | $\bar{22}$.2852759048 |
| 60.00000 | 26.0576689142 | $\bar{27}$.9423310858 |
| 70.00000 | 30.4006137332 | $\bar{31}$.5993862668 |
| 80.00000 | 34.7435585523 | $\bar{35}$.2564414477 |
| 90.00000 | 39.0865033713 | $\bar{40}$.9134966287 |
| 100.00000 | 43.4294481903 | $\bar{44}$.5705518097 |
| 200.00000 | 86.8588963807 | $\bar{87}$.1411036193 |
| 300.00000 | 130.2883445710 | $\bar{131}$.7116554290 |
| 400.00000 | 173.7177927613 | $\bar{174}$.2822072387 |
| 500.00000 | 217.1472409516 | $\bar{218}$.8527590484 |

Note: $\log e^{x+y} = \log e^{x} + \log e^{y}$. Thus, $\log e^{113.1478} = 49.139465180$.

## Five-Place Natural Logarithms.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | D. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1.00 | 0.0 0000 | 0100 | 0200 | 0300 | 0399 | 0499 | 0598 | 0698 | 0797 | 0896 | 100–99 |
| 1.01 | 0.0 0995 | 1094 | 1193 | 1292 | 1390 | 1489 | 1587 | 1686 | 1784 | 1882 | 99–98 |
| 1.02 | 0.0 1980 | 2078 | 2176 | 2274 | 2372 | 2469 | 2567 | 2664 | 2762 | 2859 | 98–97 |
| 1.03 | 0.0 2956 | 3053 | 3150 | 3247 | 3343 | 3440 | 3537 | 3633 | 3730 | 3826 | 97–96 |
| 1.04 | 0.0 3922 | 4018 | 4114 | 4210 | 4306 | 4402 | 4497 | 4593 | 4688 | 4784 | 96–95 |
| 1.05 | 0.0 4879 | 4974 | 5069 | 5164 | 5259 | 5354 | 5449 | 5543 | 5638 | 5733 | 95–94 |
| 1.06 | 0.0 5827 | 5921 | 6015 | 6110 | 6204 | 6297 | 6391 | 6485 | 6579 | 6672 | 94 |
| 1.07 | 0.0 6766 | 6859 | 6953 | 7046 | 7139 | 7232 | 7325 | 7418 | 7511 | 7603 | 93 |
| 1.08 | 0.0 7696 | 7789 | 7881 | 7973 | 8066 | 8158 | 8250 | 8342 | 8434 | 8526 | 93–92 |
| 1.09 | 0.0 8618 | 8709 | 8801 | 8893 | 8984 | 9075 | 9167 | 9258 | 9349 | 9430 | 92–91 |
| 1.10 | 0.0 9531 | 9622 | 9713 | 9803 | 9894 | 9985 | *0075 | 0165 | 0256 | 0346 | 91–90 |
| 1.11 | 0.1 0436 | 0526 | 0616 | 0706 | 0796 | 0885 | 0975 | 1065 | 1154 | 1244 | 90–89 |
| 1.12 | 0.1 1333 | 1422 | 1511 | 1600 | 1689 | 1778 | 1867 | 1956 | 2045 | 2133 | 89 |
| 1.13 | 0.1 2222 | 2310 | 2399 | 2487 | 2575 | 2663 | 2751 | 2839 | 2927 | 3015 | 88 |
| 1.14 | 0.1 3103 | 3191 | 3278 | 3366 | 3453 | 3540 | 3628 | 3715 | 3802 | 3889 | 88–87 |
| 1.15 | 0.1 3976 | 4063 | 4150 | 4237 | 4323 | 4410 | 4497 | 4583 | 4669 | 4756 | 87–86 |
| 1.16 | 0.1 4842 | 4928 | 5014 | 5100 | 5186 | 5272 | 5358 | 5444 | 5529 | 5615 | 86 |
| 1.17 | 0.1 5700 | 5786 | 5871 | 5956 | 6042 | 6127 | 6212 | 6297 | 6382 | 6467 | 85 |
| 1.18 | 0.1 6551 | 6636 | 6721 | 6805 | 6890 | 6974 | 7059 | 7143 | 7227 | 7311 | 85–84 |
| 1.19 | 0.1 7395 | 7479 | 7563 | 7647 | 7731 | 7815 | 7898 | 7982 | 8065 | 8149 | 84–83 |
| 1.20 | 0.1 8232 | 8315 | 8399 | 8482 | 8565 | 8648 | 8731 | 8814 | 8897 | 8979 | 83 |
| 1.21 | 0.1 9062 | 9145 | 9227 | 9310 | 9392 | 9474 | 9557 | 9639 | 9721 | 9803 | 83–82 |
| 1.22 | 0.1 9885 | 9967 | *0049 | 0131 | 0212 | 0294 | 0376 | 0457 | 0539 | 0620 | 82–81 |
| 1.23 | 0.2 0701 | 0783 | 0864 | 0945 | 1026 | 1107 | 1188 | 1269 | 1350 | 1430 | 81 |
| 1.24 | 0.2 1511 | 1592 | 1672 | 1753 | 1833 | 1914 | 1994 | 2074 | 2154 | 2234 | 81–80 |
| 1.25 | 0.2 2314 | 2394 | 2474 | 2554 | 2634 | 2714 | 2793 | 2873 | 2952 | 3032 | 80–79 |
| 1.26 | 0.2 3111 | 3191 | 3270 | 3349 | 3428 | 3507 | 3586 | 3665 | 3744 | 3823 | 79 |
| 1.27 | 0.2 3902 | 3980 | 4059 | 4138 | 4216 | 4295 | 4373 | 4451 | 4530 | 4608 | 79–78 |
| 1.28 | 0.2 4686 | 4764 | 4842 | 4920 | 4998 | 5076 | 5154 | 5231 | 5309 | 5387 | 78 |
| 1.29 | 0.2 5464 | 5542 | 5619 | 5697 | 5774 | 5851 | 5928 | 6005 | 6082 | 6159 | 77 |
| 1.30 | 0.2 6236 | 6313 | 6390 | 6467 | 6544 | 6620 | 6697 | 6773 | 6850 | 6926 | 77–76 |
| 1.31 | 0.2 7003 | 7079 | 7155 | 7231 | 7308 | 7384 | 7460 | 7536 | 7612 | 7687 | 76 |
| 1.32 | 0.2 7763 | 7839 | 7915 | 7990 | 8066 | 8141 | 8217 | 8292 | 8367 | 8443 | 76–75 |
| 1.33 | 0.2 8518 | 8593 | 8668 | 8743 | 8818 | 8893 | 8968 | 9043 | 9118 | 9192 | 75 |
| 1.34 | 0.2 9267 | 9342 | 9416 | 9491 | 9565 | 9639 | 9714 | 9788 | 9862 | 9936 | 75–74 |
| 1.35 | 0.3 0010 | 0085 | 0158 | 0232 | 0306 | 0380 | 0454 | 0528 | 0601 | 0675 | 74 |
| 1.36 | 0.3 0748 | 0822 | 0895 | 0969 | 1042 | 1115 | 1189 | 1262 | 1335 | 1408 | 74–73 |
| 1.37 | 0.3 1481 | 1554 | 1627 | 1700 | 1773 | 1845 | 1918 | 1991 | 2063 | 2136 | 73–72 |
| 1.38 | 0.3 2208 | 2281 | 2353 | 2426 | 2498 | 2570 | 2642 | 2714 | 2786 | 2858 | 72 |
| 1.39 | 0.3 2930 | 3002 | 3074 | 3146 | 3218 | 3289 | 3361 | 3433 | 3504 | 3576 | 72–71 |
| 1.40 | 0.3 3647 | 3719 | 3790 | 3861 | 3933 | 4004 | 4075 | 4146 | 4217 | 4288 | 71 |
| 1.41 | 0.3 4359 | 4430 | 4501 | 4572 | 4642 | 4713 | 4784 | 4854 | 4925 | 4995 | 71–70 |
| 1.42 | 0.3 5066 | 5136 | 5206 | 5277 | 5347 | 5417 | 5487 | 5557 | 5627 | 5697 | 70 |
| 1.43 | 0.3 5767 | 5837 | 5907 | 5977 | 6047 | 6116 | 6186 | 6256 | 6325 | 6395 | 70–69 |
| 1.44 | 0.3 6464 | 6534 | 6603 | 6672 | 6742 | 6811 | 6880 | 6949 | 7018 | 7087 | 69 |
| 1.45 | 0.3 7156 | 7225 | 7294 | 7363 | 7432 | 7501 | 7569 | 7638 | 7707 | 7775 | 69 |
| 1.46 | 0.3 7844 | 7912 | 7981 | 8049 | 8117 | 8186 | 8254 | 8322 | 8390 | 8458 | 68 |
| 1.47 | 0.3 8526 | 8594 | 8662 | 8730 | 8798 | 8866 | 8934 | 9001 | 9069 | 9137 | 68 |
| 1.48 | 0.3 9204 | 9272 | 9339 | 9407 | 9474 | 9541 | 9609 | 9676 | 9743 | 9810 | 68–67 |
| 1.49 | 0.3 9878 | 9945 | *0012 | 0079 | 0146 | 0213 | 0279 | 0346 | 0413 | 0480 | 67 |
| 1.50 | 0.4 0547 | 0613 | 0680 | 0746 | 0813 | 0879 | 0946 | 1012 | 1078 | 1145 | 67–66 |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |

## Five-Place Natural Logarithms.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | D. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1.50 | 0.4 0547 | 0613 | 0680 | 0746 | 0813 | 0879 | 0946 | 1012 | 1078 | 1145 | 67–66 |
| 1.51 | 0.4 1211 | 1277 | 1343 | 1409 | 1476 | 1542 | 1608 | 1673 | 1739 | 1805 | 66 |
| 1.52 | 0.4 1871 | 1937 | 2003 | 2068 | 2134 | 2199 | 2265 | 2331 | 2396 | 2461 | 66–65 |
| 1.53 | 0.4 2527 | 2592 | 2657 | 2723 | 2788 | 2853 | 2918 | 2983 | 3048 | 3113 | 65 |
| 1.54 | 0.4 3178 | 3243 | 3308 | 3373 | 3438 | 3502 | 3567 | 3632 | 3696 | 3761 | 65–64 |
| 1.55 | 0.4 3825 | 3890 | 3954 | 4019 | 4083 | 4148 | 4212 | 4276 | 4340 | 4404 | 64 |
| 1.56 | 0.4 4469 | 4533 | 4597 | 4661 | 4725 | 4789 | 4852 | 4916 | 4980 | 5044 | 64 |
| 1.57 | 0.4 5108 | 5171 | 5235 | 5298 | 5362 | 5426 | 5489 | 5552 | 5616 | 5679 | 64–63 |
| 1.58 | 0.4 5742 | 5806 | 5869 | 5932 | 5995 | 6058 | 6122 | 6185 | 6248 | 6310 | 63 |
| 1.59 | 0.4 6373 | 6436 | 6499 | 6562 | 6625 | 6687 | 6750 | 6813 | 6875 | 6938 | 63 |
| 1.60 | 0.4 7000 | 7063 | 7125 | 7188 | 7250 | 7312 | 7375 | 7437 | 7499 | 7561 | 62 |
| 1.61 | 0.4 7623 | 7686 | 7748 | 7810 | 7872 | 7933 | 7995 | 8057 | 8119 | 8181 | 62 |
| 1.62 | 0.4 8243 | 8304 | 8366 | 8428 | 8489 | 8551 | 8612 | 8674 | 8735 | 8797 | 62–61 |
| 1.63 | 0.4 8858 | 8919 | 8981 | 9042 | 9103 | 9164 | 9225 | 9287 | 9348 | 9409 | 61 |
| 1.64 | 0.4 9470 | 9531 | 9592 | 9652 | 9713 | 9774 | 9835 | 9896 | 9956 | *0017 | 61 |
| 1.65 | 0.5 0078 | 0138 | 0199 | 0259 | 0320 | 0380 | 0441 | 0501 | 0561 | 0622 | 61–60 |
| 1.66 | 0.5 0682 | 0742 | 0802 | 0862 | 0922 | 0983 | 1043 | 1103 | 1163 | 1222 | 60 |
| 1.67 | 0.5 1282 | 1342 | 1402 | 1462 | 1522 | 1581 | 1641 | 1701 | 1760 | 1820 | 60 |
| 1.68 | 0.5 1879 | 1939 | 1998 | 2058 | 2117 | 2177 | 2236 | 2295 | 2354 | 2414 | 60–59 |
| 1.69 | 0.5 2473 | 2532 | 2591 | 2650 | 2709 | 2768 | 2827 | 2886 | 2945 | 3004 | 59 |
| 1.70 | 0.5 3063 | 3122 | 3180 | 3239 | 3298 | 3357 | 3415 | 3474 | 3532 | 3591 | 59–58 |
| 1.71 | 0.5 3649 | 3708 | 3766 | 3825 | 3883 | 3941 | 4000 | 4058 | 4116 | 4174 | 58 |
| 1.72 | 0.5 4232 | 4291 | 4349 | 4407 | 4465 | 4523 | 4581 | 4639 | 4696 | 4754 | 58 |
| 1.73 | 0.5 4812 | 4870 | 4928 | 4985 | 5043 | 5101 | 5158 | 5216 | 5274 | 5331 | 58–57 |
| 1.74 | 0.5 5389 | 5446 | 5503 | 5561 | 5618 | 5675 | 5733 | 5790 | 5847 | 5904 | 57 |
| 1.75 | 0.5 5962 | 6019 | 6076 | 6133 | 6190 | 6247 | 6304 | 6361 | 6418 | 6475 | 57 |
| 1.76 | 0.5 6531 | 6588 | 6645 | 6702 | 6758 | 6815 | 6872 | 6928 | 6985 | 7041 | 57 |
| 1.77 | 0.5 7098 | 7154 | 7211 | 7267 | 7324 | 7380 | 7436 | 7493 | 7549 | 7605 | 56 |
| 1.78 | 0.5 7661 | 7718 | 7774 | 7830 | 7886 | 7942 | 7998 | 8054 | 8110 | 8166 | 56 |
| 1.79 | 0.5 8222 | 8277 | 8333 | 8389 | 8445 | 8501 | 8556 | 8612 | 8667 | 8723 | 56 |
| 1.80 | 0.5 8779 | 8834 | 8890 | 8945 | 9001 | 9056 | 9111 | 9167 | 9222 | 9277 | 56–55 |
| 1.81 | 0.5 9333 | 9388 | 9443 | 9498 | 9553 | 9609 | 9664 | 9719 | 9774 | 9829 | 55 |
| 1.82 | 0.5 9884 | 9939 | 9993 | *0048 | 0103 | 0158 | 0213 | 0268 | 0322 | 0377 | 55 |
| 1.83 | 0.6 0432 | 0486 | 0541 | 0595 | 0650 | 0704 | 0759 | 0813 | 0868 | 0922 | 55–54 |
| 1.84 | 0.6 0977 | 1031 | 1085 | 1139 | 1194 | 1248 | 1302 | 1356 | 1410 | 1464 | 54 |
| 1.85 | 0.6 1519 | 1573 | 1627 | 1681 | 1735 | 1788 | 1842 | 1896 | 1950 | 2004 | 54 |
| 1.86 | 0.6 2058 | 2111 | 2165 | 2219 | 2272 | 2326 | 2380 | 2433 | 2487 | 2540 | 54–53 |
| 1.87 | 0.6 2594 | 2647 | 2701 | 2754 | 2808 | 2861 | 2914 | 2967 | 3021 | 3074 | 53 |
| 1.88 | 0.6 3127 | 3180 | 3234 | 3287 | 3340 | 3393 | 3446 | 3499 | 3552 | 3605 | 53 |
| 1.89 | 0.6 3658 | 3711 | 3763 | 3816 | 3869 | 3922 | 3975 | 4027 | 4080 | 4133 | 53 |
| 1.90 | 0.6 4185 | 4238 | 4291 | 4343 | 4396 | 4448 | 4501 | 4553 | 4606 | 4658 | 53–52 |
| 1.91 | 0.6 4710 | 4763 | 4815 | 4867 | 4920 | 4972 | 5024 | 5076 | 5128 | 5180 | 52 |
| 1.92 | 0.6 5233 | 5285 | 5337 | 5389 | 5441 | 5493 | 5545 | 5596 | 5648 | 5700 | 52 |
| 1.93 | 0.6 5752 | 5804 | 5856 | 5907 | 5959 | 6011 | 6062 | 6114 | 6166 | 6217 | 52 |
| 1.94 | 0.6 6269 | 6320 | 6372 | 6423 | 6475 | 6526 | 6578 | 6629 | 6680 | 6732 | 52–51 |
| 1.95 | 0.6 6783 | 6834 | 6885 | 6937 | 6988 | 7039 | 7090 | 7141 | 7192 | 7243 | 51 |
| 1.96 | 0.6 7294 | 7345 | 7396 | 7447 | 7498 | 7549 | 7600 | 7651 | 7702 | 7753 | 51 |
| 1.97 | 0.6 7803 | 7854 | 7905 | 7956 | 8006 | 8057 | 8107 | 8158 | 8209 | 8259 | 51 |
| 1.98 | 0.6 8310 | 8360 | 8411 | 8461 | 8512 | 8562 | 8612 | 8663 | 8713 | 8763 | 50 |
| 1.99 | 0.6 8813 | 8864 | 8914 | 8964 | 9014 | 9064 | 9115 | 9165 | 9215 | 9265 | 50 |
| 2.00 | 0.6 9315 | 9365 | 9415 | 9465 | 9515 | 9564 | 9614 | 9664 | 9714 | 9764 | 50 |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |

## Five-Place Natural Logarithms.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | D. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **2.00** | 0.6 9315 | 9365 | 9415 | 9465 | 9515 | 9564 | 9614 | 9664 | 9714 | 9764 | 50 |
| 2.01 | 0.6 9813 | 9863 | 9913 | 9963 | *0012 | 0062 | 0112 | 0161 | 0211 | 0260 | 50 |
| 2.02 | 0.7 0310 | 0359 | 0409 | 0458 | 0508 | 0557 | 0606 | 0656 | 0705 | 0754 | 49 |
| 2.03 | 0.7 0804 | 0853 | 0902 | 0951 | 1000 | 1050 | 1099 | 1148 | 1197 | 1246 | 49 |
| 2.04 | 0.7 1295 | 1344 | 1393 | 1442 | 1491 | 1540 | 1589 | 1638 | 1686 | 1735 | 49 |
| 2.05 | 0.7 1784 | 1833 | 1881 | 1930 | 1979 | 2028 | 2076 | 2125 | 2173 | 2222 | 49 |
| 2.06 | 0.7 2271 | 2319 | 2368 | 2416 | 2465 | 2513 | 2561 | 2610 | 2658 | 2707 | 49–48 |
| 2.07 | 0.7 2755 | 2803 | 2851 | 2900 | 2948 | 2996 | 3044 | 3092 | 3141 | 3189 | 48 |
| 2.08 | 0.7 3237 | 3285 | 3333 | 3381 | 3429 | 3477 | 3525 | 3573 | 3621 | 3669 | 48 |
| 2.09 | 0.7 3716 | 3764 | 3812 | 3860 | 3908 | 3955 | 4003 | 4051 | 4098 | 4146 | 48 |
| **2.10** | 0.7 4194 | 4241 | 4289 | 4336 | 4384 | 4432 | 4479 | 4527 | 4574 | 4621 | 48–47 |
| 2.11 | 0.7 4669 | 4716 | 4764 | 4811 | 4858 | 4905 | 4953 | 5000 | 5047 | 5094 | 47 |
| 2.12 | 0.7 5142 | 5189 | 5236 | 5283 | 5330 | 5377 | 5424 | 5471 | 5518 | 5565 | 47 |
| 2.13 | 0.7 5612 | 5659 | 5706 | 5753 | 5800 | 5847 | 5893 | 5940 | 5987 | 6034 | 47 |
| 2.14 | 0.7 6081 | 6127 | 6174 | 6221 | 6267 | 6314 | 6361 | 6407 | 6454 | 6500 | 47 |
| 2.15 | 0.7 6547 | 6593 | 6640 | 6686 | 6733 | 6779 | 6825 | 6872 | 6918 | 6965 | 47–46 |
| 2.16 | 0.7 7011 | 7057 | 7103 | 7150 | 7196 | 7242 | 7288 | 7334 | 7381 | 7427 | 46 |
| 2.17 | 0.7 7473 | 7519 | 7565 | 7611 | 7657 | 7703 | 7749 | 7795 | 7841 | 7887 | 46 |
| 2.18 | 0.7 7932 | 7978 | 8024 | 8070 | 8116 | 8162 | 8207 | 8253 | 8299 | 8344 | 46 |
| 2.19 | 0.7 8390 | 8436 | 8481 | 8527 | 8573 | 8618 | 8664 | 8709 | 8755 | 8800 | 46–45 |
| **2.20** | 0.7 8846 | 8891 | 8937 | 8982 | 9027 | 9073 | 9118 | 9163 | 9209 | 9254 | 45 |
| 2.21 | 0.7 9299 | 9344 | 9390 | 9435 | 9480 | 9525 | 9570 | 9615 | 9661 | 9706 | 45 |
| 2.22 | 0.7 9751 | 9796 | 9841 | 9886 | 9931 | 9976 | *0021 | 0066 | 0110 | 0155 | 45 |
| 2.23 | 0.8 0200 | 0245 | 0290 | 0335 | 0379 | 0424 | 0469 | 0514 | 0558 | 0603 | 45 |
| 2.24 | 0.8 0648 | 0692 | 0737 | 0781 | 0826 | 0871 | 0915 | 0960 | 1004 | 1049 | 45–44 |
| 2.25 | 0.8 1093 | 1137 | 1182 | 1226 | 1271 | 1315 | 1359 | 1404 | 1448 | 1492 | 44 |
| 2.26 | 0.8 1536 | 1581 | 1625 | 1669 | 1713 | 1757 | 1802 | 1846 | 1890 | 1934 | 44 |
| 2.27 | 0.8 1978 | 2022 | 2066 | 2110 | 2154 | 2198 | 2242 | 2286 | 2330 | 2374 | 44 |
| 2.28 | 0.8 2418 | 2461 | 2505 | 2549 | 2593 | 2637 | 2680 | 2724 | 2768 | 2812 | 44 |
| 2.29 | 0.8 2855 | 2899 | 2942 | 2986 | 3030 | 3073 | 3117 | 3160 | 3204 | 3247 | 44–43 |
| **2.30** | 0.8 3291 | 3334 | 3378 | 3421 | 3465 | 3508 | 3551 | 3595 | 3638 | 3681 | 43 |
| 2.31 | 0.8 3725 | 3768 | 3811 | 3855 | 3898 | 3941 | 3984 | 4027 | 4070 | 4114 | 43 |
| 2.32 | 0.8 4157 | 4200 | 4243 | 4286 | 4329 | 4372 | 4415 | 4458 | 4501 | 4544 | 43 |
| 2.33 | 0.8 4587 | 4630 | 4673 | 4715 | 4758 | 4801 | 4844 | 4887 | 4930 | 4972 | 43 |
| 2.34 | 0.8 5015 | 5058 | 5101 | 5143 | 5186 | 5229 | 5271 | 5314 | 5356 | 5399 | 43 |
| 2.35 | 0.8 5442 | 5484 | 5527 | 5569 | 5612 | 5654 | 5697 | 5739 | 5781 | 5824 | 43–42 |
| 2.36 | 0.8 5866 | 5909 | 5951 | 5993 | 6036 | 6078 | 6120 | 6162 | 6205 | 6247 | 42 |
| 2.37 | 0.8 6289 | 6331 | 6373 | 6415 | 6458 | 6500 | 6542 | 6584 | 6626 | 6668 | 42 |
| 2.38 | 0.8 6710 | 6752 | 6794 | 6836 | 6878 | 6920 | 6962 | 7004 | 7046 | 7087 | 42 |
| 2.39 | 0.8 7129 | 7171 | 7213 | 7255 | 7297 | 7338 | 7380 | 7422 | 7464 | 7505 | 42 |
| **2.40** | 0.8 7547 | 7589 | 7630 | 7672 | 7713 | 7755 | 7797 | 7838 | 7880 | 7921 | 42 |
| 2.41 | 0.8 7963 | 8004 | 8046 | 8087 | 8129 | 8170 | 8211 | 8253 | 8294 | 8335 | 41 |
| 2.42 | 0.8 8377 | 8418 | 8459 | 8501 | 8542 | 8583 | 8624 | 8666 | 8707 | 8748 | 41 |
| 2.43 | 0.8 8789 | 8830 | 8871 | 8913 | 8954 | 8995 | 9036 | 9077 | 9118 | 9159 | 41 |
| 2.44 | 0.8 9200 | 9241 | 9282 | 9323 | 9364 | 9405 | 9445 | 9486 | 9527 | 9568 | 41 |
| 2.45 | 0.8 9609 | 9650 | 9690 | 9731 | 9772 | 9813 | 9853 | 9894 | 9935 | 9975 | 41 |
| 2.46 | 0.9 0016 | 0057 | 0097 | 0138 | 0179 | 0219 | 0260 | 0300 | 0341 | 0381 | 41–40 |
| 2.47 | 0.9 0422 | 0462 | 0503 | 0543 | 0584 | 0624 | 0664 | 0705 | 0745 | 0786 | 40 |
| 2.48 | 0.9 0826 | 0866 | 0906 | 0947 | 0987 | 1027 | 1067 | 1108 | 1148 | 1188 | 40 |
| 2.49 | 0.9 1228 | 1268 | 1309 | 1349 | 1389 | 1429 | 1469 | 1509 | 1549 | 1589 | 40 |
| **2.50** | 0.9 1629 | 1669 | 1709 | 1749 | 1789 | 1829 | 1869 | 1909 | 1949 | 1988 | 40 |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |

## Five-Place Natural Logarithms.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | D. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2.50 | 0.9 1629 | 1669 | 1709 | 1749 | 1789 | 1829 | 1869 | 1909 | 1949 | 1988 | 40 |
| 2.51 | 0.9 2028 | 2068 | 2108 | 2148 | 2188 | 2227 | 2267 | 2307 | 2346 | 2386 | 40 |
| 2.52 | 0.9 2426 | 2466 | 2505 | 2545 | 2584 | 2624 | 2664 | 2703 | 2743 | 2782 | 40 |
| 2.53 | 0.9 2822 | 2861 | 2901 | 2940 | 2980 | 3019 | 3059 | 3098 | 3138 | 3177 | 40–39 |
| 2.54 | 0.9 3216 | 3256 | 3295 | 3334 | 3374 | 3413 | 3452 | 3492 | 3531 | 3570 | 39 |
| 2.55 | 0.9 3609 | 3649 | 3688 | 3727 | 3766 | 3805 | 3844 | 3883 | 3923 | 3962 | 39 |
| 2.56 | 0.9 4001 | 4040 | 4079 | 4118 | 4157 | 4196 | 4235 | 4274 | 4313 | 4352 | 39 |
| 2.57 | 0.9 4391 | 4429 | 4468 | 4507 | 4546 | 4585 | 4624 | 4663 | 4701 | 4740 | 39 |
| 2.58 | 0.9 4779 | 4818 | 4856 | 4895 | 4934 | 4973 | 5011 | 5050 | 5089 | 5127 | 39 |
| 2.59 | 0.9 5166 | 5204 | 5243 | 5282 | 5320 | 5359 | 5397 | 5436 | 5474 | 5513 | 39–38 |
| 2.60 | 0.9 5551 | 5590 | 5628 | 5666 | 5705 | 5743 | 5782 | 5820 | 5858 | 5897 | 38 |
| 2.61 | 0.9 5935 | 5973 | 6012 | 6050 | 6088 | 6126 | 6165 | 6203 | 6241 | 6279 | 38 |
| 2.62 | 0.9 6317 | 6356 | 6394 | 6432 | 6470 | 6508 | 6546 | 6584 | 6622 | 6660 | 38 |
| 2.63 | 0.9 6698 | 6736 | 6774 | 6812 | 6850 | 6888 | 6926 | 6964 | 7002 | 7040 | 38 |
| 2.64 | 0.9 7078 | 7116 | 7154 | 7191 | 7229 | 7267 | 7305 | 7343 | 7380 | 7418 | 38 |
| 2.65 | 0.9 7456 | 7494 | 7531 | 7569 | 7607 | 7644 | 7682 | 7720 | 7757 | 7795 | 38 |
| 2.66 | 0.9 7833 | 7870 | 7908 | 7945 | 7983 | 8020 | 8058 | 8095 | 8133 | 8170 | 38–37 |
| 2.67 | 0.9 8208 | 8245 | 8283 | 8320 | 8358 | 8395 | 8432 | 8470 | 8507 | 8544 | 37 |
| 2.68 | 0.9 8582 | 8619 | 8656 | 8694 | 8731 | 8768 | 8805 | 8843 | 8880 | 8917 | 37 |
| 2.69 | 0.9 8954 | 8991 | 9028 | 9066 | 9103 | 9140 | 9177 | 9214 | 9251 | 9288 | 37 |
| 2.70 | 0.9 9325 | 9362 | 9399 | 9436 | 9473 | 9510 | 9547 | 9584 | 9621 | 9658 | 37 |
| 2.71 | 0.9 9695 | 9732 | 9769 | 9806 | 9842 | 9879 | 9916 | 9953 | 9990 | *0026 | 37 |
| 2.72 | 1.0 0063 | 0100 | 0137 | 0173 | 0210 | 0247 | 0284 | 0320 | 0357 | 0394 | 37 |
| 2.73 | 1.0 0430 | 0467 | 0503 | 0540 | 0577 | 0613 | 0650 | 0686 | 0723 | 0759 | 37 |
| 2.74 | 1.0 0796 | 0832 | 0869 | 0905 | 0942 | 0978 | 1015 | 1051 | 1087 | 1124 | 36 |
| 2.75 | 1.0 1160 | 1196 | 1233 | 1269 | 1305 | 1342 | 1378 | 1414 | 1451 | 1487 | 36 |
| 2.76 | 1.0 1523 | 1559 | 1596 | 1632 | 1668 | 1704 | 1740 | 1776 | 1813 | 1849 | 36 |
| 2.77 | 1.0 1885 | 1921 | 1957 | 1993 | 2029 | 2065 | 2101 | 2137 | 2173 | 2209 | 36 |
| 2.78 | 1.0 2245 | 2281 | 2317 | 2353 | 2389 | 2425 | 2461 | 2497 | 2532 | 2568 | 36 |
| 2.79 | 1.0 2604 | 2640 | 2676 | 2712 | 2747 | 2783 | 2819 | 2855 | 2890 | 2926 | 36 |
| 2.80 | 1.0 2962 | 2998 | 3033 | 3069 | 3105 | 3140 | 3176 | 3212 | 3247 | 3283 | 36 |
| 2.81 | 1.0 3318 | 3354 | 3390 | 3425 | 3461 | 3496 | 3532 | 3567 | 3603 | 3638 | 36–35 |
| 2.82 | 1.0 3674 | 3709 | 3745 | 3780 | 3815 | 3851 | 3886 | 3922 | 3957 | 3992 | 35 |
| 2.83 | 1.0 4028 | 4063 | 4098 | 4134 | 4169 | 4204 | 4239 | 4275 | 4310 | 4345 | 35 |
| 2.84 | 1.0 4380 | 4416 | 4451 | 4486 | 4521 | 4556 | 4591 | 4627 | 4662 | 4697 | 35 |
| 2.85 | 1.0 4732 | 4767 | 4802 | 4837 | 4872 | 4907 | 4942 | 4977 | 5012 | 5047 | 35 |
| 2.86 | 1.0 5082 | 5117 | 5152 | 5187 | 5222 | 5257 | 5292 | 5327 | 5361 | 5396 | 35 |
| 2.87 | 1.0 5431 | 5466 | 5501 | 5536 | 5570 | 5605 | 5640 | 5675 | 5710 | 5744 | 35 |
| 2.88 | 1.0 5779 | 5814 | 5848 | 5883 | 5918 | 5952 | 5987 | 6022 | 6056 | 6091 | 35 |
| 2.89 | 1.0 6126 | 6160 | 6195 | 6229 | 6264 | 6299 | 6333 | 6368 | 6402 | 6437 | 35–34 |
| 2.90 | 1.0 6471 | 6506 | 6540 | 6574 | 6609 | 6643 | 6678 | 6712 | 6747 | 6781 | 34 |
| 2.91 | 1.0 6815 | 6850 | 6884 | 6918 | 6953 | 6987 | 7021 | 7056 | 7090 | 7124 | 34 |
| 2.92 | 1.0 7158 | 7193 | 7227 | 7261 | 7295 | 7329 | 7364 | 7398 | 7432 | 7466 | 34 |
| 2.93 | 1.0 7500 | 7534 | 7568 | 7603 | 7637 | 7671 | 7705 | 7739 | 7773 | 7807 | 34 |
| 2.94 | 1.0 7841 | 7875 | 7909 | 7943 | 7977 | 8011 | 8045 | 8079 | 8113 | 8147 | 34 |
| 2.95 | 1.0 8181 | 8214 | 8248 | 8282 | 8316 | 8350 | 8384 | 8418 | 8451 | 8485 | 34 |
| 2.96 | 1.0 8519 | 8553 | 8586 | 8620 | 8654 | 8688 | 8721 | 8755 | 8789 | 8823 | 34 |
| 2.97 | 1.0 8856 | 8890 | 8924 | 8957 | 8991 | 9024 | 9058 | 9092 | 9125 | 9159 | 34 |
| 2.98 | 1.0 9192 | 9226 | 9259 | 9293 | 9326 | 9360 | 9393 | 9427 | 9460 | 9494 | 34–33 |
| 2.99 | 1.0 9527 | 9561 | 9594 | 9628 | 9661 | 9694 | 9728 | 9761 | 9795 | 9828 | 33 |
| 3.00 | 1.0 9861 | 9895 | 9928 | 9961 | 9994 | *0028 | 0061 | 0094 | 0128 | 0161 | 33 |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |

## Five-Place Natural Logarithms.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | D. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 3.00 | 1.0 9861 | 9895 | 9928 | 9961 | 9994 | *0028 | 0061 | 0094 | 0128 | 0161 | 33 |
| 3.01 | 1.1 0194 | 0227 | 0260 | 0294 | 0327 | 0360 | 0393 | 0426 | 0459 | 0493 | 33 |
| 3.02 | 1.1 0526 | 0559 | 0592 | 0625 | 0658 | 0691 | 0724 | 0757 | 0790 | 0823 | 33 |
| 3.03 | 1.1 0856 | 0889 | 0922 | 0955 | 0988 | 1021 | 1054 | 1087 | 1120 | 1153 | 33 |
| 3.04 | 1.1 1186 | 1219 | 1252 | 1284 | 1317 | 1350 | 1383 | 1416 | 1449 | 1481 | 33 |
| 3.05 | 1.1 1514 | 1547 | 1580 | 1612 | 1645 | 1678 | 1711 | 1743 | 1776 | 1809 | 33 |
| 3.06 | 1.1 1841 | 1874 | 1907 | 1939 | 1972 | 2005 | 2037 | 2070 | 2103 | 2135 | 33 |
| 3.07 | 1.1 2168 | 2200 | 2233 | 2265 | 2298 | 2330 | 2363 | 2396 | 2428 | 2460 | 33–32 |
| 3.08 | 1.1 2493 | 2525 | 2558 | 2590 | 2623 | 2655 | 2688 | 2720 | 2752 | 2785 | 32 |
| 3.09 | 1.1 2817 | 2849 | 2882 | 2914 | 2946 | 2979 | 3011 | 3043 | 3076 | 3108 | 32 |
| 3.10 | 1.1 3140 | 3172 | 3205 | 3237 | 3269 | 3301 | 3334 | 3366 | 3398 | 3430 | 32 |
| 3.11 | 1.1 3462 | 3494 | 3527 | 3559 | 3591 | 3623 | 3655 | 3687 | 3719 | 3751 | 32 |
| 3.12 | 1.1 3783 | 3815 | 3847 | 3879 | 3911 | 3943 | 3975 | 4007 | 4039 | 4071 | 32 |
| 3.13 | 1.1 4103 | 4135 | 4167 | 4199 | 4231 | 4263 | 4295 | 4327 | 4359 | 4390 | 32 |
| 3.14 | 1.1 4422 | 4454 | 4486 | 4518 | 4550 | 4581 | 4613 | 4645 | 4677 | 4708 | 32 |
| 3.15 | 1.1 4740 | 4772 | 4804 | 4835 | 4867 | 4899 | 4931 | 4962 | 4994 | 5026 | 32 |
| 3.16 | 1.1 5057 | 5089 | 5120 | 5152 | 5184 | 5215 | 5247 | 5278 | 5310 | 5342 | 32 |
| 3.17 | 1.1 5373 | 5405 | 5436 | 5468 | 5499 | 5531 | 5562 | 5594 | 5625 | 5657 | 32–31 |
| 3.18 | 1.1 5688 | 5720 | 5751 | 5782 | 5814 | 5845 | 5877 | 5908 | 5939 | 5971 | 31 |
| 3.19 | 1.1 6002 | 6033 | 6065 | 6096 | 6127 | 6159 | 6190 | 6221 | 6253 | 6284 | 31 |
| 3.20 | 1.1 6315 | 6346 | 6378 | 6409 | 6440 | 6471 | 6502 | 6534 | 6565 | 6596 | 31 |
| 3.21 | 1.1 6627 | 6658 | 6689 | 6721 | 6752 | 6783 | 6814 | 6845 | 6876 | 6907 | 31 |
| 3.22 | 1.1 6938 | 6969 | 7000 | 7031 | 7062 | 7093 | 7124 | 7155 | 7186 | 7217 | 31 |
| 3.23 | 1.1 7248 | 7279 | 7310 | 7341 | 7372 | 7403 | 7434 | 7465 | 7496 | 7526 | 31 |
| 3.24 | 1.1 7557 | 7588 | 7619 | 7650 | 7681 | 7712 | 7742 | 7773 | 7804 | 7835 | 31 |
| 3.25 | 1.1 7865 | 7896 | 7927 | 7958 | 7989 | 8019 | 8050 | 8081 | 8111 | 8142 | 31 |
| 3.26 | 1.1 8173 | 8203 | 8234 | 8265 | 8295 | 8326 | 8357 | 8387 | 8418 | 8448 | 31 |
| 3.27 | 1.1 8479 | 8510 | 8540 | 8571 | 8601 | 8632 | 8662 | 8693 | 8723 | 8754 | 31–30 |
| 3.28 | 1.1 8784 | 8815 | 8845 | 8876 | 8906 | 8937 | 8967 | 8998 | 9028 | 9058 | 30 |
| 3.29 | 1.1 9089 | 9119 | 9150 | 9180 | 9210 | 9241 | 9271 | 9301 | 9332 | 9362 | 30 |
| 3.30 | 1.1 9392 | 9423 | 9453 | 9483 | 9513 | 9544 | 9574 | 9604 | 9634 | 9665 | 30 |
| 3.31 | 1.1 9695 | 9725 | 9755 | 9785 | 9816 | 9846 | 9876 | 9906 | 9936 | 9966 | 30 |
| 3.32 | 1.1 9996 | *0027 | 0057 | 0087 | 0117 | 0147 | 0177 | 0207 | 0237 | 0267 | 30 |
| 3.33 | 1.2 0297 | 0327 | 0357 | 0387 | 0417 | 0447 | 0477 | 0507 | 0537 | 0567 | 30 |
| 3.34 | 1.2 0597 | 0627 | 0657 | 0687 | 0717 | 0747 | 0777 | 0806 | 0836 | 0866 | 30 |
| 3.35 | 1.2 0896 | 0926 | 0956 | 0986 | 1015 | 1045 | 1075 | 1105 | 1135 | 1164 | 30 |
| 3.36 | 1.2 1194 | 1224 | 1254 | 1283 | 1313 | 1343 | 1373 | 1402 | 1432 | 1462 | 30 |
| 3.37 | 1.2 1491 | 1521 | 1551 | 1580 | 1610 | 1640 | 1669 | 1699 | 1728 | 1758 | 30 |
| 3.38 | 1.2 1788 | 1817 | 1847 | 1876 | 1906 | 1935 | 1965 | 1994 | 2024 | 2053 | 30 |
| 3.39 | 1.2 2083 | 2112 | 2142 | 2171 | 2201 | 2230 | 2260 | 2289 | 2319 | 2348 | 29 |
| 3.40 | 1.2 2378 | 2407 | 2436 | 2466 | 2495 | 2524 | 2554 | 2583 | 2613 | 2642 | 29 |
| 3.41 | 1.2 2671 | 2701 | 2730 | 2759 | 2788 | 2818 | 2847 | 2876 | 2906 | 2935 | 29 |
| 3.42 | 1.2 2964 | 2993 | 3023 | 3052 | 3081 | 3110 | 3139 | 3169 | 3198 | 3227 | 29 |
| 3.43 | 1.2 3256 | 3285 | 3314 | 3343 | 3373 | 3402 | 3431 | 3460 | 3489 | 3518 | 29 |
| 3.44 | 1.2 3547 | 3576 | 3605 | 3634 | 3663 | 3692 | 3721 | 3750 | 3779 | 3808 | 29 |
| 3.45 | 1.2 3837 | 3866 | 3895 | 3924 | 3953 | 3982 | 4011 | 4040 | 4069 | 4098 | 29 |
| 3.46 | 1.2 4127 | 4156 | 4185 | 4214 | 4242 | 4271 | 4300 | 4329 | 4358 | 4387 | 29 |
| 3.47 | 1.2 4415 | 4444 | 4473 | 4502 | 4531 | 4559 | 4588 | 4617 | 4646 | 4674 | 29 |
| 3.48 | 1.2 4703 | 4732 | 4761 | 4789 | 4818 | 4847 | 4875 | 4904 | 4933 | 4962 | 29 |
| 3.49 | 1.2 4990 | 5019 | 5047 | 5076 | 5105 | 5133 | 5162 | 5191 | 5219 | 5248 | 29 |
| 3.50 | 1.2 5276 | 5305 | 5333 | 5362 | 5391 | 5419 | 5448 | 5476 | 5505 | 5533 | 29–28 |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |

## Five-Place Natural Logarithms.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | D. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **3.50** | 1.2 5276 | 5305 | 5333 | 5362 | 5391 | 5419 | 5448 | 5476 | 5505 | 5533 | 29–28 |
| 3.51 | 1.2 5562 | 5590 | 5619 | 5647 | 5675 | 5704 | 5732 | 5761 | 5789 | 5818 | 28 |
| 3.52 | 1.2 5846 | 5875 | 5903 | 5931 | 5960 | 5988 | 6016 | 6045 | 6073 | 6101 | 28 |
| 3.53 | 1.2 6130 | 6158 | 6186 | 6215 | 6243 | 6271 | 6300 | 6328 | 6356 | 6384 | 28 |
| 3.54 | 1.2 6413 | 6441 | 6469 | 6497 | 6526 | 6554 | 6582 | 6610 | 6638 | 6667 | 28 |
| 3.55 | 1.2 6695 | 6723 | 6751 | 6779 | 6807 | 6836 | 6864 | 6892 | 6920 | 6948 | 28 |
| 3.56 | 1.2 6976 | 7004 | 7032 | 7060 | 7088 | 7116 | 7144 | 7172 | 7201 | 7229 | 28 |
| 3.57 | 1.2 7257 | 7285 | 7313 | 7341 | 7369 | 7397 | 7424 | 7452 | 7480 | 7508 | 28 |
| 3.58 | 1.2 7536 | 7564 | 7592 | 7620 | 7648 | 7676 | 7704 | 7732 | 7759 | 7787 | 28 |
| 3.59 | 1.2 7815 | 7843 | 7871 | 7899 | 7927 | 7954 | 7982 | 8010 | 8038 | 8066 | 28 |
| **3.60** | 1.2 8093 | 8121 | 8149 | 8177 | 8204 | 8232 | 8260 | 8288 | 8315 | 8343 | 28 |
| 3.61 | 1.2 8371 | 8398 | 8426 | 8454 | 8482 | 8509 | 8537 | 8564 | 8592 | 8620 | 28 |
| 3.62 | 1.2 8647 | 8675 | 8703 | 8730 | 8758 | 8785 | 8813 | 8841 | 8868 | 8896 | 28 |
| 3.63 | 1.2 8923 | 8951 | 8978 | 9006 | 9033 | 9061 | 9088 | 9116 | 9143 | 9171 | 28–27 |
| 3.64 | 1.2 9198 | 9226 | 9253 | 9281 | 9308 | 9336 | 9363 | 9390 | 9418 | 9445 | 27 |
| 3.65 | 1.2 9473 | 9500 | 9527 | 9555 | 9582 | 9610 | 9637 | 9664 | 9692 | 9719 | 27 |
| 3.66 | 1.2 9746 | 9774 | 9801 | 9828 | 9856 | 9883 | 9910 | 9937 | 9965 | 9992 | 27 |
| 3.67 | 1.3 0019 | 0046 | 0074 | 0101 | 0128 | 0155 | 0183 | 0210 | 0237 | 0264 | 27 |
| 3.68 | 1.3 0291 | 0318 | 0346 | 0373 | 0400 | 0427 | 0454 | 0481 | 0508 | 0536 | 27 |
| 3.69 | 1.3 0563 | 0590 | 0617 | 0644 | 0671 | 0698 | 0725 | 0752 | 0779 | 0806 | 27 |
| **3.70** | 1.3 0833 | 0860 | 0887 | 0914 | 0941 | 0968 | 0995 | 1022 | 1049 | 1076 | 27 |
| 3.71 | 1.3 1103 | 1130 | 1157 | 1184 | 1211 | 1238 | 1265 | 1292 | 1319 | 1345 | 27 |
| 3.72 | 1.3 1372 | 1399 | 1426 | 1453 | 1480 | 1507 | 1534 | 1560 | 1587 | 1614 | 27 |
| 3.73 | 1.3 1641 | 1668 | 1694 | 1721 | 1748 | 1775 | 1802 | 1828 | 1855 | 1882 | 27 |
| 3.74 | 1.3 1909 | 1935 | 1962 | 1989 | 2015 | 2042 | 2069 | 2096 | 2122 | 2149 | 27 |
| 3.75 | 1.3 2176 | 2202 | 2229 | 2256 | 2282 | 2309 | 2335 | 2362 | 2389 | 2415 | 27 |
| 3.76 | 1.3 2442 | 2468 | 2495 | 2522 | 2548 | 2575 | 2601 | 2628 | 2654 | 2681 | 27 |
| 3.77 | 1.3 2708 | 2734 | 2761 | 2787 | 2814 | 2840 | 2867 | 2893 | 2919 | 2946 | 27–26 |
| 3.78 | 1.3 2972 | 2999 | 3025 | 3052 | 3078 | 3105 | 3131 | 3157 | 3184 | 3210 | 26 |
| 3.79 | 1.3 3237 | 3263 | 3289 | 3316 | 3342 | 3368 | 3395 | 3421 | 3447 | 3474 | 26 |
| **3.80** | 1.3 3500 | 3526 | 3553 | 3579 | 3605 | 3632 | 3658 | 3684 | 3710 | 3737 | 26 |
| 3.81 | 1.3 3763 | 3789 | 3815 | 3842 | 3868 | 3894 | 3920 | 3946 | 3973 | 3999 | 26 |
| 3.82 | 1.3 4025 | 4051 | 4077 | 4104 | 4130 | 4156 | 4182 | 4208 | 4234 | 4260 | 26 |
| 3.83 | 1.3 4286 | 4313 | 4339 | 4365 | 4391 | 4417 | 4443 | 4469 | 4495 | 4521 | 26 |
| 3.84 | 1.3 4547 | 4573 | 4599 | 4625 | 4651 | 4677 | 4703 | 4729 | 4755 | 4781 | 26 |
| 3.85 | 1.3 4807 | 4833 | 4859 | 4885 | 4911 | 4937 | 4963 | 4989 | 5015 | 5041 | 26 |
| 3.86 | 1.3 5067 | 5093 | 5119 | 5144 | 5170 | 5196 | 5222 | 5248 | 5274 | 5300 | 26 |
| 3.87 | 1.3 5325 | 5351 | 5377 | 5403 | 5429 | 5455 | 5480 | 5506 | 5532 | 5558 | 26 |
| 3.88 | 1.3 5584 | 5609 | 5635 | 5661 | 5687 | 5712 | 5738 | 5764 | 5789 | 5815 | 26 |
| 3.89 | 1.3 5841 | 5867 | 5892 | 5918 | 5944 | 5969 | 5995 | 6021 | 6046 | 6072 | 26 |
| **3.90** | 1.3 6098 | 6123 | 6149 | 6175 | 6200 | 6226 | 6251 | 6277 | 6303 | 6328 | 26 |
| 3.91 | 1.3 6354 | 6379 | 6405 | 6430 | 6456 | 6481 | 6507 | 6533 | 6558 | 6584 | 26 |
| 3.92 | 1.3 6609 | 6635 | 6660 | 6686 | 6711 | 6737 | 6762 | 6788 | 6813 | 6838 | 26–25 |
| 3.93 | 1.3 6864 | 6889 | 6915 | 6940 | 6966 | 6991 | 7016 | 7042 | 7067 | 7093 | 25 |
| 3.94 | 1.3 7118 | 7143 | 7169 | 7194 | 7220 | 7245 | 7270 | 7296 | 7321 | 7346 | 25 |
| 3.95 | 1.3 7372 | 7397 | 7422 | 7447 | 7473 | 7498 | 7523 | 7549 | 7574 | 7599 | 25 |
| 3.96 | 1.3 7624 | 7650 | 7675 | 7700 | 7725 | 7751 | 7776 | 7801 | 7826 | 7851 | 25 |
| 3.97 | 1.3 7877 | 7902 | 7927 | 7952 | 7977 | 8002 | 8028 | 8053 | 8078 | 8103 | 25 |
| 3.98 | 1.3 8128 | 8153 | 8178 | 8204 | 8229 | 8254 | 8279 | 8304 | 8329 | 8354 | 25 |
| 3.99 | 1.3 8379 | 8404 | 8429 | 8454 | 8479 | 8504 | 8529 | 8554 | 8579 | 8604 | 25 |
| **4.00** | 1.3 8629 | 8654 | 8679 | 8704 | 8729 | 8754 | 8779 | 8804 | 8829 | 8854 | 25 |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |

## Five-Place Natural Logarithms.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | D. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 4.00 | 1.3 8629 | 8654 | 8679 | 8704 | 8729 | 8754 | 8779 | 8804 | 8829 | 8854 | 25 |
| 4.01 | 1.3 8879 | 8904 | 8929 | 8954 | 8979 | 9004 | 9029 | 9054 | 9078 | 9103 | 25 |
| 4.02 | 1.3 9128 | 9153 | 9178 | 9203 | 9228 | 9252 | 9277 | 9302 | 9327 | 9352 | 25 |
| 4.03 | 1.3 9377 | 9401 | 9426 | 9451 | 9476 | 9501 | 9525 | 9550 | 9575 | 9600 | 25 |
| 4.04 | 1.3 9624 | 9649 | 9674 | 9699 | 9723 | 9748 | 9773 | 9798 | 9822 | 9847 | 25 |
| 4.05 | 1.3 9872 | 9896 | 9921 | 9946 | 9970 | 9995 | *0020 | 0044 | 0069 | 0094 | 25 |
| 4.06 | 1.4 0118 | 0143 | 0168 | 0192 | 0217 | 0241 | 0266 | 0291 | 0315 | 0340 | 25 |
| 4.07 | 1.4 0364 | 0389 | 0413 | 0438 | 0463 | 0487 | 0512 | 0536 | 0561 | 0585 | 25 |
| 4.08 | 1.4 0610 | 0634 | 0659 | 0683 | 0708 | 0732 | 0757 | 0781 | 0806 | 0830 | 25–24 |
| 4.09 | 1.4 0854 | 0879 | 0903 | 0928 | 0952 | 0977 | 1001 | 1025 | 1050 | 1074 | 24 |
| 4.10 | 1.4 1099 | 1123 | 1147 | 1172 | 1196 | 1221 | 1245 | 1269 | 1294 | 1318 | 24 |
| 4.11 | 1.4 1342 | 1367 | 1391 | 1415 | 1440 | 1464 | 1488 | 1512 | 1537 | 1561 | 24 |
| 4.12 | 1.4 1585 | 1610 | 1634 | 1658 | 1682 | 1707 | 1731 | 1755 | 1779 | 1804 | 24 |
| 4.13 | 1.4 1828 | 1852 | 1876 | 1900 | 1925 | 1949 | 1973 | 1997 | 2021 | 2045 | 24 |
| 4.14 | 1.4 2070 | 2094 | 2118 | 2142 | 2166 | 2190 | 2214 | 2239 | 2263 | 2287 | 24 |
| 4.15 | 1.4 2311 | 2335 | 2359 | 2383 | 2407 | 2431 | 2455 | 2479 | 2503 | 2527 | 24 |
| 4.16 | 1.4 2552 | 2576 | 2600 | 2624 | 2648 | 2672 | 2696 | 2720 | 2744 | 2768 | 24 |
| 4.17 | 1.4 2792 | 2816 | 2840 | 2864 | 2887 | 2911 | 2935 | 2959 | 2983 | 3007 | 24 |
| 4.18 | 1.4 3031 | 3055 | 3079 | 3103 | 3127 | 3151 | 3175 | 3198 | 3222 | 3246 | 24 |
| 4.19 | 1.4 3270 | 3294 | 3318 | 3342 | 3365 | 3389 | 3413 | 3437 | 3461 | 3485 | 24 |
| 4.20 | 1.4 3508 | 3532 | 3556 | 3580 | 3604 | 3627 | 3651 | 3675 | 3699 | 3723 | 24 |
| 4.21 | 1.4 3746 | 3770 | 3794 | 3817 | 3841 | 3865 | 3889 | 3912 | 3936 | 3960 | 24 |
| 4.22 | 1.4 3984 | 4007 | 4031 | 4055 | 4078 | 4102 | 4126 | 4149 | 4173 | 4197 | 24 |
| 4.23 | 1.4 4220 | 4244 | 4267 | 4291 | 4315 | 4338 | 4362 | 4386 | 4409 | 4433 | 24 |
| 4.24 | 1.4 4456 | 4480 | 4503 | 4527 | 4551 | 4574 | 4598 | 4621 | 4645 | 4668 | 24 |
| 4.25 | 1.4 4692 | 4715 | 4739 | 4762 | 4786 | 4809 | 4833 | 4856 | 4880 | 4903 | 24–23 |
| 4.26 | 1.4 4927 | 4950 | 4974 | 4997 | 5021 | 5044 | 5068 | 5091 | 5115 | 5138 | 23 |
| 4.27 | 1.4 5161 | 5185 | 5208 | 5232 | 5255 | 5278 | 5302 | 5325 | 5349 | 5372 | 23 |
| 4.28 | 1.4 5395 | 5419 | 5442 | 5465 | 5489 | 5512 | 5535 | 5559 | 5582 | 5605 | 23 |
| 4.29 | 1.4 5629 | 5652 | 5675 | 5699 | 5722 | 5745 | 5768 | 5792 | 5815 | 5838 | 23 |
| 4.30 | 1.4 5862 | 5885 | 5908 | 5931 | 5954 | 5978 | 6001 | 6024 | 6047 | 6071 | 23 |
| 4.31 | 1.4 6094 | 6117 | 6140 | 6163 | 6187 | 6210 | 6233 | 6256 | 6279 | 6302 | 23 |
| 4.32 | 1.4 6326 | 6349 | 6372 | 6395 | 6418 | 6441 | 6464 | 6487 | 6511 | 6534 | 23 |
| 4.33 | 1.4 6557 | 6580 | 6603 | 6626 | 6649 | 6672 | 6695 | 6718 | 6741 | 6764 | 23 |
| 4.34 | 1.4 6787 | 6810 | 6834 | 6857 | 6880 | 6903 | 6926 | 6949 | 6972 | 6995 | 23 |
| 4.35 | 1.4 7018 | 7041 | 7064 | 7087 | 7109 | 7132 | 7155 | 7178 | 7201 | 7224 | 23 |
| 4.36 | 1.4 7247 | 7270 | 7293 | 7316 | 7339 | 7362 | 7385 | 7408 | 7431 | 7453 | 23 |
| 4.37 | 1.4 7476 | 7499 | 7522 | 7545 | 7568 | 7591 | 7614 | 7636 | 7659 | 7682 | 23 |
| 4.38 | 1.4 7705 | 7728 | 7751 | 7773 | 7796 | 7819 | 7842 | 7865 | 7887 | 7910 | 23 |
| 4.39 | 1.4 7933 | 7956 | 7978 | 8001 | 8024 | 8047 | 8070 | 8092 | 8115 | 8138 | 23 |
| 4.40 | 1.4 8160 | 8183 | 8206 | 8229 | 8251 | 8274 | 8297 | 8319 | 8342 | 8365 | 23 |
| 4.41 | 1.4 8387 | 8410 | 8433 | 8455 | 8478 | 8501 | 8523 | 8546 | 8569 | 8591 | 23 |
| 4.42 | 1.4 8614 | 8637 | 8659 | 8682 | 8704 | 8727 | 8750 | 8772 | 8795 | 8817 | 23 |
| 4.43 | 1.4 8840 | 8863 | 8885 | 8908 | 8930 | 8953 | 8975 | 8998 | 9020 | 9043 | 23 |
| 4.44 | 1.4 9065 | 9088 | 9110 | 9133 | 9155 | 9178 | 9200 | 9223 | 9245 | 9268 | 23 |
| 4.45 | 1.4 9290 | 9313 | 9335 | 9358 | 9380 | 9403 | 9425 | 9448 | 9470 | 9492 | 23–22 |
| 4.46 | 1.4 9515 | 9537 | 9560 | 9582 | 9605 | 9627 | 9649 | 9672 | 9694 | 9716 | 22 |
| 4.47 | 1.4 9739 | 9761 | 9784 | 9806 | 9828 | 9851 | 9873 | 9895 | 9918 | 9940 | 22 |
| 4.48 | 1.4 9962 | 9985 | *0007 | 0029 | 0052 | 0074 | 0096 | 0118 | 0141 | 0163 | 22 |
| 4.49 | 1.5 0185 | 0208 | 0230 | 0252 | 0274 | 0297 | 0319 | 0341 | 0363 | 0386 | 22 |
| 4.50 | 1.5 0408 | 0430 | 0452 | 0474 | 0497 | 0519 | 0541 | 0563 | 0585 | 0608 | 22 |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |

## Five-Place Natural Logarithms.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | D. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 4.50 | 1.5 0408 | 0430 | 0452 | 0474 | 0497 | 0519 | 0541 | 0563 | 0585 | 0608 | 22 |
| 4.51 | 1.5 0630 | 0652 | 0674 | 0696 | 0718 | 0741 | 0763 | 0785 | 0807 | 0829 | 22 |
| 4.52 | 1.5 0851 | 0873 | 0895 | 0918 | 0940 | 0962 | 0984 | 1006 | 1028 | 1050 | 22 |
| 4.53 | 1.5 1072 | 1094 | 1116 | 1138 | 1160 | 1183 | 1205 | 1227 | 1249 | 1271 | 22 |
| 4.54 | 1.5 1293 | 1315 | 1337 | 1359 | 1381 | 1403 | 1425 | 1447 | 1469 | 1491 | 22 |
| 4.55 | 1.5 1513 | 1535 | 1557 | 1579 | 1601 | 1623 | 1645 | 1666 | 1688 | 1710 | 22 |
| 4.56 | 1.5 1732 | 1754 | 1776 | 1798 | 1820 | 1842 | 1864 | 1886 | 1908 | 1929 | 22 |
| 4.57 | 1.5 1951 | 1973 | 1995 | 2017 | 2039 | 2061 | 2083 | 2104 | 2126 | 2148 | 22 |
| 4.58 | 1.5 2170 | 2192 | 2214 | 2235 | 2257 | 2279 | 2301 | 2323 | 2344 | 2366 | 22 |
| 4.59 | 1.5 2388 | 2410 | 2432 | 2453 | 2475 | 2497 | 2519 | 2540 | 2562 | 2584 | 22 |
| 4.60 | 1.5 2606 | 2627 | 2649 | 2671 | 2693 | 2714 | 2736 | 2758 | 2779 | 2801 | 22 |
| 4.61 | 1.5 2823 | 2844 | 2866 | 2888 | 2910 | 2931 | 2953 | 2975 | 2996 | 3018 | 22 |
| 4.62 | 1.5 3039 | 3061 | 3083 | 3104 | 3126 | 3148 | 3169 | 3191 | 3212 | 3234 | 22 |
| 4.63 | 1.5 3256 | 3277 | 3299 | 3320 | 3342 | 3364 | 3385 | 3407 | 3428 | 3450 | 22 |
| 4.64 | 1.5 3471 | 3493 | 3515 | 3536 | 3558 | 3579 | 3601 | 3622 | 3644 | 3665 | 22 |
| 4.65 | 1.5 3687 | 3708 | 3730 | 3751 | 3773 | 3794 | 3816 | 3837 | 3859 | 3880 | 22–21 |
| 4.66 | 1.5 3902 | 3923 | 3944 | 3966 | 3987 | 4009 | 4030 | 4052 | 4073 | 4094 | 21 |
| 4.67 | 1.5 4116 | 4137 | 4159 | 4180 | 4202 | 4223 | 4244 | 4266 | 4287 | 4308 | 21 |
| 4.68 | 1.5 4330 | 4351 | 4373 | 4394 | 4415 | 4437 | 4458 | 4479 | 4501 | 4522 | 21 |
| 4.69 | 1.5 4543 | 4565 | 4586 | 4607 | 4629 | 4650 | 4671 | 4692 | 4714 | 4735 | 21 |
| 4.70 | 1.5 4756 | 4778 | 4799 | 4820 | 4841 | 4863 | 4884 | 4905 | 4926 | 4948 | 21 |
| 4.71 | 1.5 4969 | 4990 | 5011 | 5032 | 5054 | 5075 | 5096 | 5117 | 5138 | 5160 | 21 |
| 4.72 | 1.5 5181 | 5202 | 5223 | 5244 | 5266 | 5287 | 5308 | 5329 | 5350 | 5371 | 21 |
| 4.73 | 1.5 5393 | 5414 | 5435 | 5456 | 5477 | 5498 | 5519 | 5540 | 5562 | 5583 | 21 |
| 4.74 | 1.5 5604 | 5625 | 5646 | 5667 | 5688 | 5709 | 5730 | 5751 | 5772 | 5793 | 21 |
| 4.75 | 1.5 5814 | 5836 | 5857 | 5878 | 5899 | 5920 | 5941 | 5962 | 5983 | 6004 | 21 |
| 4.76 | 1.5 6025 | 6046 | 6067 | 6088 | 6109 | 6130 | 6151 | 6172 | 6193 | 6214 | 21 |
| 4.77 | 1.5 6235 | 6256 | 6277 | 6298 | 6318 | 6339 | 6360 | 6381 | 6402 | 6423 | 21 |
| 4.78 | 1.5 6444 | 6465 | 6486 | 6507 | 6528 | 6549 | 6569 | 6590 | 6611 | 6632 | 21 |
| 4.79 | 1.5 6653 | 6674 | 6695 | 6716 | 6737 | 6757 | 6778 | 6799 | 6820 | 6841 | 21 |
| 4.80 | 1.5 6862 | 6882 | 6903 | 6924 | 6945 | 6966 | 6987 | 7007 | 7028 | 7049 | 21 |
| 4.81 | 1.5 7070 | 7090 | 7111 | 7132 | 7153 | 7174 | 7194 | 7215 | 7236 | 7257 | 21 |
| 4.82 | 1.5 7277 | 7298 | 7319 | 7340 | 7360 | 7381 | 7402 | 7423 | 7443 | 7464 | 21 |
| 4.83 | 1.5 7485 | 7505 | 7526 | 7547 | 7567 | 7588 | 7609 | 7629 | 7650 | 7671 | 21 |
| 4.84 | 1.5 7691 | 7712 | 7733 | 7753 | 7774 | 7795 | 7815 | 7836 | 7857 | 7877 | 21 |
| 4.85 | 1.5 7898 | 7918 | 7939 | 7960 | 7980 | 8001 | 8022 | 8042 | 8063 | 8083 | 21 |
| 4.86 | 1.5 8104 | 8124 | 8145 | 8166 | 8186 | 8207 | 8227 | 8248 | 8268 | 8289 | 21 |
| 4.87 | 1.5 8309 | 8330 | 8350 | 8371 | 8391 | 8412 | 8433 | 8453 | 8474 | 8494 | 21–20 |
| 4.88 | 1.5 8515 | 8535 | 8555 | 8576 | 8596 | 8617 | 8637 | 8658 | 8678 | 8699 | 20 |
| 4.89 | 1.5 8719 | 8740 | 8760 | 8781 | 8801 | 8821 | 8842 | 8862 | 8883 | 8903 | 20 |
| 4.90 | 1.5 8924 | 8944 | 8964 | 8985 | 9005 | 9026 | 9046 | 9066 | 9087 | 9107 | 20 |
| 4.91 | 1.5 9127 | 9148 | 9168 | 9188 | 9209 | 9229 | 9250 | 9270 | 9290 | 9311 | 20 |
| 4.92 | 1.5 9331 | 9351 | 9371 | 9392 | 9412 | 9432 | 9453 | 9473 | 9493 | 9514 | 20 |
| 4.93 | 1.5 9534 | 9554 | 9574 | 9595 | 9615 | 9635 | 9656 | 9676 | 9696 | 9716 | 20 |
| 4.94 | 1.5 9737 | 9757 | 9777 | 9797 | 9817 | 9838 | 9858 | 9878 | 9898 | 9919 | 20 |
| 4.95 | 1.5 9939 | 9959 | 9979 | 9999 | *0020 | 0040 | 0060 | 0080 | 0100 | 0120 | 20 |
| 4.96 | 1.6 0141 | 0161 | 0181 | 0201 | 0221 | 0241 | 0261 | 0282 | 0302 | 0322 | 20 |
| 4.97 | 1.6 0342 | 0362 | 0382 | 0402 | 0422 | 0443 | 0463 | 0483 | 0503 | 0523 | 20 |
| 4.98 | 1.6 0543 | 0563 | 0583 | 0603 | 0623 | 0643 | 0663 | 0683 | 0704 | 0724 | 20 |
| 4.99 | 1.6 0744 | 0764 | 0784 | 0804 | 0824 | 0844 | 0864 | 0884 | 0904 | 0924 | 20 |
| 5.00 | 1.6 0944 | 0964 | 0984 | 1004 | 1024 | 1044 | 1064 | 1084 | 1104 | 1124 | 20 |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |

## Five-Place Natural Logarithms.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | D. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5.0 | 1.6 0944 | 1144 | 1343 | 1542 | 1741 | 1939 | 2137 | 2334 | 2531 | 2728 | 200–196 |
| 5.1 | 1.6 2924 | 3120 | 3315 | 3511 | 3705 | 3900 | 4094 | 4287 | 4481 | 4673 | 196–192 |
| 5.2 | 1.6 4866 | 5058 | 5250 | 5441 | 5632 | 5823 | 6013 | 6203 | 6393 | 6582 | 192–189 |
| 5.3 | 1.6 6771 | 6959 | 7147 | 7335 | 7523 | 7710 | 7896 | 8083 | 8269 | 8455 | 189–185 |
| 5.4 | 1.6 8640 | 8825 | 9010 | 9194 | 9378 | 9562 | 9745 | 9928 | *0111 | 0293 | 185–182 |
| 5.5 | 1.7 0475 | 0656 | 0838 | 1019 | 1199 | 1380 | 1560 | 1740 | 1919 | 2098 | 182–179 |
| 5.6 | 1.7 2277 | 2455 | 2633 | 2811 | 2988 | 3166 | 3342 | 3519 | 3695 | 3871 | 178–175 |
| 5.7 | 1.7 4047 | 4222 | 4397 | 4572 | 4746 | 4920 | 5094 | 5267 | 5440 | 5613 | 175–173 |
| 5.8 | 1.7 5786 | 5958 | 6130 | 6302 | 6473 | 6644 | 6815 | 6985 | 7156 | 7326 | 172–170 |
| 5.9 | 1.7 7495 | 7665 | 7834 | 8002 | 8171 | 8339 | 8507 | 8675 | 8842 | 9009 | 169–167 |
| 6.0 | 1.7 9176 | 9342 | 9509 | 9675 | 9840 | *0006 | 0171 | 0336 | 0500 | 0665 | 167–164 |
| 6.1 | 1.8 0829 | 0993 | 1156 | 1319 | 1482 | 1645 | 1808 | 1970 | 2132 | 2294 | 164–161 |
| 6.2 | 1.8 2455 | 2616 | 2777 | 2938 | 3098 | 3258 | 3418 | 3578 | 3737 | 3896 | 161–159 |
| 6.3 | 1.8 4055 | 4214 | 4372 | 4530 | 4688 | 4845 | 5003 | 5160 | 5317 | 5473 | 159–156 |
| 6.4 | 1.8 5630 | 5786 | 5942 | 6097 | 6253 | 6408 | 6563 | 6718 | 6872 | 7026 | 156–154 |
| 6.5 | 1.8 7180 | 7334 | 7487 | 7641 | 7794 | 7947 | 8099 | 8251 | 8403 | 8555 | 154–152 |
| 6.6 | 1.8 8707 | 8858 | 9010 | 9160 | 9311 | 9462 | 9612 | 9762 | 9912 | *0061 | 151–149 |
| 6.7 | 1.9 0211 | 0360 | 0509 | 0658 | 0806 | 0954 | 1102 | 1250 | 1398 | 1545 | 149–147 |
| 6.8 | 1.9 1692 | 1839 | 1986 | 2132 | 2279 | 2425 | 2571 | 2716 | 2862 | 3007 | 147–145 |
| 6.9 | 1.9 3152 | 3297 | 3442 | 3586 | 3730 | 3874 | 4018 | 4162 | 4305 | 4448 | 145–143 |
| 7.0 | 1.9 4591 | 4734 | 4876 | 5019 | 5161 | 5303 | 5445 | 5586 | 5727 | 5869 | 143–141 |
| 7.1 | 1.9 6009 | 6150 | 6291 | 6431 | 6571 | 6711 | 6851 | 6991 | 7130 | 7269 | 141–139 |
| 7.2 | 1.9 7408 | 7547 | 7685 | 7824 | 7962 | 8100 | 8238 | 8376 | 8513 | 8650 | 139–137 |
| 7.3 | 1.9 8787 | 8924 | 9061 | 9198 | 9334 | 9470 | 9606 | 9742 | 9877 | *0013 | 137–135 |
| 7.4 | 2.0 0148 | 0283 | 0418 | 0553 | 0687 | 0821 | 0956 | 1089 | 1223 | 1357 | 135–133 |
| 7.5 | 2.0 1490 | 1624 | 1757 | 1890 | 2022 | 2155 | 2287 | 2419 | 2551 | 2683 | 133–132 |
| 7.6 | 2.0 2815 | 2946 | 3078 | 3209 | 3340 | 3471 | 3601 | 3732 | 3862 | 3992 | 131–130 |
| 7.7 | 2.0 4122 | 4252 | 4381 | 4511 | 4640 | 4769 | 4898 | 5027 | 5156 | 5284 | 130–128 |
| 7.8 | 2.0 5412 | 5540 | 5668 | 5796 | 5924 | 6051 | 6179 | 6306 | 6433 | 6560 | 128–127 |
| 7.9 | 2.0 6686 | 6813 | 6939 | 7065 | 7191 | 7317 | 7443 | 7568 | 7694 | 7819 | 127–125 |
| 8.0 | 2.0 7944 | 8069 | 8194 | 8318 | 8443 | 8567 | 8691 | 8815 | 8939 | 9063 | 125–124 |
| 8.1 | 2.0 9186 | 9310 | 9433 | 9556 | 9679 | 9802 | 9924 | *0047 | 0169 | 0291 | 123–122 |
| 8.2 | 2.1 0413 | 0535 | 0657 | 0779 | 0900 | 1021 | 1142 | 1263 | 1384 | 1505 | 122–121 |
| 8.3 | 2.1 1626 | 1746 | 1866 | 1986 | 2106 | 2226 | 2346 | 2465 | 2585 | 2704 | 120–119 |
| 8.4 | 2.1 2823 | 2942 | 3061 | 3180 | 3298 | 3417 | 3535 | 3653 | 3771 | 3889 | 119–118 |
| 8.5 | 2.1 4007 | 4124 | 4242 | 4359 | 4476 | 4593 | 4710 | 4827 | 4943 | 5060 | 118–116 |
| 8.6 | 2.1 5176 | 5292 | 5409 | 5524 | 5640 | 5756 | 5871 | 5987 | 6102 | 6217 | 116–115 |
| 8.7 | 2.1 6332 | 6447 | 6562 | 6677 | 6791 | 6905 | 7020 | 7134 | 7248 | 7361 | 115–114 |
| 8.8 | 2.1 7475 | 7589 | 7702 | 7816 | 7929 | 8042 | 8155 | 8267 | 8380 | 8493 | 114–112 |
| 8.9 | 2.1 8605 | 8717 | 8830 | 8942 | 9054 | 9165 | 9277 | 9389 | 9500 | 9611 | 112–111 |
| 9.0 | 2.1 9722 | 9834 | 9944 | *0055 | 0166 | 0276 | 0387 | 0497 | 0607 | 0717 | 111–110 |
| 9.1 | 2.2 0827 | 0937 | 1047 | 1157 | 1266 | 1375 | 1485 | 1594 | 1703 | 1812 | 110–109 |
| 9.2 | 2.2 1920 | 2029 | 2138 | 2246 | 2354 | 2462 | 2570 | 2678 | 2786 | 2894 | 109–108 |
| 9.3 | 2.2 3001 | 3109 | 3216 | 3324 | 3431 | 3538 | 3645 | 3751 | 3858 | 3965 | 107–106 |
| 9.4 | 2.2 4071 | 4177 | 4284 | 4390 | 4496 | 4601 | 4707 | 4813 | 4918 | 5024 | 106–105 |
| 9.5 | 2.2 5129 | 5234 | 5339 | 5444 | 5549 | 5654 | 5759 | 5863 | 5968 | 6072 | 105–104 |
| 9.6 | 2.2 6176 | 6280 | 6384 | 6488 | 6592 | 6696 | 6799 | 6903 | 7006 | 7109 | 104–103 |
| 9.7 | 2.2 7213 | 7316 | 7419 | 7521 | 7624 | 7727 | 7829 | 7932 | 8034 | 8136 | 103–102 |
| 9.8 | 2.2 8238 | 8340 | 8442 | 8544 | 8646 | 8747 | 8849 | 8950 | 9051 | 9152 | 102–101 |
| 9.9 | 2.2 9253 | 9354 | 9455 | 9556 | 9657 | 9757 | 9858 | 9958 | *0058 | 0158 | 101–100 |
| 10.0 | 2.3 0259 | 0358 | 0458 | 0558 | 0658 | 0757 | 0857 | 0956 | 1055 | 1154 | 100–99 |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |

**The Natural Logarithms (each increased by 10.) of Numbers between 0.00 and 0.99.**

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | | 5.395 | 6.088 | 6.493 | 6.781 | 7.004 | 7.187 | 7.341 | 7.474 | 7.592 |
| 0.1 | 7.697 | 7.793 | 7.880 | 7.960 | 8.034 | 8.103 | 8.167 | 8.228 | 8.285 | 8.339 |
| 0.2 | 8.391 | 8.439 | 8.486 | 8.530 | 8.573 | 8.614 | 8.653 | 8.691 | 8.727 | 8.762 |
| 0.3 | 8.796 | 8.829 | 8.861 | 8.891 | 8.921 | 8.950 | 8.978 | 9.006 | 9.032 | 9.058 |
| 0.4 | 9.084 | 9.108 | 9.132 | 9.156 | 9.179 | 9.201 | 9.223 | 9.245 | 9.266 | 9.287 |
| 0.5 | 9.307 | 9.327 | 9.346 | 9.365 | 9.384 | 9.402 | 9.420 | 9.438 | 9.455 | 9.472 |
| 0.6 | 9.489 | 9.506 | 9.522 | 9.538 | 9.554 | 9.569 | 9.584 | 9.600 | 9.614 | 9.629 |
| 0.7 | 9.643 | 9.658 | 9.671 | 9.685 | 9.699 | 9.712 | 9.726 | 9.739 | 9.752 | 9.764 |
| 0.8 | 9.777 | 9.789 | 9.802 | 9.814 | 9.826 | 9.837 | 9.849 | 9.861 | 9.872 | 9.883 |
| 0.9 | 9.895 | 9.906 | 9.917 | 9.927 | 9.938 | 9.949 | 9.959 | 9.970 | 9.980 | 9.990 |

Note : $\log_e x = \log_{10} x \cdot \log_e 10 = (2.30259) \log_{10} x$.

**The Natural Logarithms of Whole Numbers from 10 to 209.**

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2.3026 | 3979 | 4849 | 5649 | 6391 | 7080 | 7726 | 8332 | 8904 | 9444 |
| 2 | 2.9957 | *0445 | 0910 | 1355 | 1781 | 2189 | 2581 | 2958 | 3322 | 3673 |
| 3 | 3.4012 | 4340 | 4657 | 4965 | 5264 | 5553 | 5835 | 6109 | 6376 | 6636 |
| 4 | 3.6889 | 7136 | 7377 | 7612 | 7842 | 8067 | 8286 | 8501 | 8712 | 8918 |
| 5 | 3.9120 | 9318 | 9512 | 9703 | 9890 | *0073 | 0254 | 0431 | 0604 | 0775 |
| 6 | 4.0943 | 1109 | 1271 | 1431 | 1589 | 1744 | 1897 | 2047 | 2195 | 2341 |
| 7 | 4.2485 | 2627 | 2767 | 2905 | 3041 | 3175 | 3307 | 3438 | 3567 | 3694 |
| 8 | 4.3820 | 3944 | 4067 | 4188 | 4308 | 4427 | 4543 | 4659 | 4773 | 4886 |
| 9 | 4.4998 | 5109 | 5218 | 5326 | 5433 | 5539 | 5643 | 5747 | 5850 | 5951 |
| 10 | 4.6052 | 6151 | 6250 | 6347 | 6444 | 6540 | 6634 | 6728 | 6821 | 6913 |
| 11 | 4.7005 | 7095 | 7185 | 7274 | 7362 | 7449 | 7536 | 7622 | 7707 | 7791 |
| 12 | 4.7875 | 7958 | 8040 | 8122 | 8203 | 8283 | 8363 | 8442 | 8520 | 8598 |
| 13 | 4.8675 | 8752 | 8828 | 8903 | 8978 | 9053 | 9127 | 9200 | 9273 | 9345 |
| 14 | 4.9416 | 9488 | 9558 | 9628 | 9698 | 9767 | 9836 | 9904 | 9972 | *0039 |
| 15 | 5.0106 | 0173 | 0239 | 0304 | 0370 | 0434 | 0499 | 0562 | 0626 | 0689 |
| 16 | 5.0752 | 0814 | 0876 | 0938 | 0999 | 1059 | 1120 | 1180 | 1240 | 1299 |
| 17 | 5.1358 | 1417 | 1475 | 1533 | 1591 | 1648 | 1705 | 1762 | 1818 | 1874 |
| 18 | 5.1930 | 1985 | 2040 | 2095 | 2149 | 2204 | 2257 | 2311 | 2364 | 2417 |
| 19 | 5.2470 | 2523 | 2575 | 2627 | 2679 | 2730 | 2781 | 2832 | 2883 | 2933 |
| 20 | 5.2983 | 3033 | 3083 | 3132 | 3181 | 3230 | 3279 | 3327 | 3375 | 3423 |

Note : $\log_e 10 = 2.30258509$. $\log_e 100 = 4.60517019$.

## The Common Logarithms of $\Gamma(n)$ for Values of $n$ between 1 and 2.

$$\Gamma(n) = \int_0^\infty x^{n-1} \cdot e^{-x} dx = \int_0^1 \left[\log \frac{1}{x}\right]^{n-1} dx.$$

| $n$ | $\log_{10}\Gamma(n)$ | $n$ | $\log_{10}\Gamma(n)$ | $n$ | $\log_{10}\Gamma(n)$ | $n$ | $\log_{10}\Gamma(n)$ | $n$ | $\log_{10}\Gamma(n)$ |
|---|---|---|---|---|---|---|---|---|---|
| 1.01 | $\bar{1}.9975$ | 1.21 | $\bar{1}.9617$ | 1.41 | $\bar{1}.9478$ | 1.61 | $\bar{1}.9517$ | 1.81 | $\bar{1}.9704$ |
| 1.02 | $\bar{1}.9951$ | 1.22 | $\bar{1}.9605$ | 1.42 | $\bar{1}.9476$ | 1.62 | $\bar{1}.9523$ | 1.82 | $\bar{1}.9717$ |
| 1.03 | $\bar{1}.9928$ | 1.23 | $\bar{1}.9594$ | 1.43 | $\bar{1}.9475$ | 1.63 | $\bar{1}.9529$ | 1.83 | $\bar{1}.9730$ |
| 1.04 | $\bar{1}.9905$ | 1.24 | $\bar{1}.9583$ | 1.44 | $\bar{1}.9473$ | 1.64 | $\bar{1}.9536$ | 1.84 | $\bar{1}.9743$ |
| 1.05 | $\bar{1}.9883$ | 1.25 | $\bar{1}.9573$ | 1.45 | $\bar{1}.9473$ | 1.65 | $\bar{1}.9543$ | 1.85 | $\bar{1}.9757$ |
| 1.06 | $\bar{1}.9862$ | 1.26 | $\bar{1}.9564$ | 1.46 | $\bar{1}.9472$ | 1.66 | $\bar{1}.9550$ | 1.86 | $\bar{1}.9771$ |
| 1.07 | $\bar{1}.9841$ | 1.27 | $\bar{1}.9554$ | 1.47 | $\bar{1}.9473$ | 1.67 | $\bar{1}.9558$ | 1.87 | $\bar{1}.9786$ |
| 1.08 | $\bar{1}.9821$ | 1.28 | $\bar{1}.9546$ | 1.48 | $\bar{1}.9473$ | 1.68 | $\bar{1}.9566$ | 1.88 | $\bar{1}.9800$ |
| 1.09 | $\bar{1}.9802$ | 1.29 | $\bar{1}.9538$ | 1.49 | $\bar{1}.9474$ | 1.69 | $\bar{1}.9575$ | 1.89 | $\bar{1}.9815$ |
| 1.10 | $\bar{1}.9783$ | 1.30 | $\bar{1}.9530$ | 1.50 | $\bar{1}.9475$ | 1.70 | $\bar{1}.9584$ | 1.90 | $\bar{1}.9831$ |
| 1.11 | $\bar{1}.9765$ | 1.31 | $\bar{1}.9523$ | 1.51 | $\bar{1}.9477$ | 1.71 | $\bar{1}.9593$ | 1.91 | $\bar{1}.9846$ |
| 1.12 | $\bar{1}.9748$ | 1.32 | $\bar{1}.9516$ | 1.52 | $\bar{1}.9479$ | 1.72 | $\bar{1}.9603$ | 1.92 | $\bar{1}.9862$ |
| 1.13 | $\bar{1}.9731$ | 1.33 | $\bar{1}.9510$ | 1.53 | $\bar{1}.9482$ | 1.73 | $\bar{1}.9613$ | 1.93 | $\bar{1}.9878$ |
| 1.14 | $\bar{1}.9715$ | 1.34 | $\bar{1}.9505$ | 1.54 | $\bar{1}.9485$ | 1.74 | $\bar{1}.9623$ | 1.94 | $\bar{1}.9895$ |
| 1.15 | $\bar{1}.9699$ | 1.35 | $\bar{1}.9500$ | 1.55 | $\bar{1}.9488$ | 1.75 | $\bar{1}.9633$ | 1.95 | $\bar{1}.9912$ |
| 1.16 | $\bar{1}.9684$ | 1.36 | $\bar{1}.9495$ | 1.56 | $\bar{1}.9492$ | 1.76 | $\bar{1}.9644$ | 1.96 | $\bar{1}.9929$ |
| 1.17 | $\bar{1}.9669$ | 1.37 | $\bar{1}.9491$ | 1.57 | $\bar{1}.9496$ | 1.77 | $\bar{1}.9656$ | 1.97 | $\bar{1}.9946$ |
| 1.18 | $\bar{1}.9655$ | 1.38 | $\bar{1}.9487$ | 1.58 | $\bar{1}.9501$ | 1.78 | $\bar{1}.9667$ | 1.98 | $\bar{1}.9964$ |
| 1.19 | $\bar{1}.9642$ | 1.39 | $\bar{1}.9483$ | 1.59 | $\bar{1}.9506$ | 1.79 | $\bar{1}.9679$ | 1.99 | $\bar{1}.9982$ |
| 1.20 | $\bar{1}.9629$ | 1.40 | $\bar{1}.9481$ | 1.60 | $\bar{1}.9511$ | 1.80 | $\bar{1}.9691$ | 2.00 | 0.0000 |

$$\Gamma(z+1) = z \cdot \Gamma(z), \quad z > 1.$$

## NATURAL TRIGONOMETRIC FUNCTIONS.

| Angle. | Sin. | Csc. | Tan. | Ctn. | Sec. | Cos. | |
|---|---|---|---|---|---|---|---|
| 0° | 0.000 | ∞ | 0.000 | ∞ | 1.000 | 1.000 | 90° |
| 1 | 0.017 | 57.30 | 0.017 | 57.29 | 1.000 | 1.000 | 89 |
| 2 | 0.035 | 28.65 | 0.035 | 28.64 | 1.001 | 0.999 | 88 |
| 3 | 0.052 | 19.11 | 0.052 | 19.08 | 1.001 | 0.999 | 87 |
| 4 | 0.070 | 14.34 | 0.070 | 14.30 | 1.002 | 0.998 | 86 |
| 5° | 0.087 | 11.47 | 0.087 | 11.43 | 1.004 | 0.996 | 85° |
| 6 | 0.105 | 9.567 | 0.105 | 9.514 | 1.006 | 0.995 | 84 |
| 7 | 0.122 | 8.206 | 0.123 | 8.144 | 1.008 | 0.993 | 83 |
| 8 | 0.139 | 7.185 | 0.141 | 7.115 | 1.010 | 0.990 | 82 |
| 9 | 0.156 | 6.392 | 0.158 | 6.314 | 1.012 | 0.988 | 81 |
| 10° | 0.174 | 5.759 | 0.176 | 5.671 | 1.015 | 0.985 | 80° |
| 11 | 0.191 | 5.241 | 0.194 | 5.145 | 1.019 | 0.982 | 79 |
| 12 | 0.208 | 4.810 | 0.213 | 4.705 | 1.022 | 0.978 | 78 |
| 13 | 0.225 | 4.445 | 0.231 | 4.331 | 1.026 | 0.974 | 77 |
| 14 | 0.242 | 4.134 | 0.249 | 4.011 | 1.031 | 0.970 | 76 |
| 15° | 0.259 | 3.864 | 0.268 | 3.732 | 1.035 | 0.966 | 75° |
| 16 | 0.276 | 3.628 | 0.287 | 3.487 | 1.040 | 0.961 | 74 |
| 17 | 0.292 | 3.420 | 0.306 | 3.271 | 1.046 | 0.956 | 73 |
| 18 | 0.309 | 3.236 | 0.325 | 3.078 | 1.051 | 0.951 | 72 |
| 19 | 0.326 | 3.072 | 0.344 | 2.904 | 1.058 | 0.946 | 71 |
| 20° | 0.342 | 2.924 | 0.364 | 2.747 | 1.064 | 0.940 | 70° |
| 21 | 0.358 | 2.790 | 0.384 | 2.605 | 1.071 | 0.934 | 69 |
| 22 | 0.375 | 2.669 | 0.404 | 2.475 | 1.079 | 0.927 | 68 |
| 23 | 0.391 | 2.559 | 0.424 | 2.356 | 1.086 | 0.921 | 67 |
| 24 | 0.407 | 2.459 | 0.445 | 2.246 | 1.095 | 0.914 | 66 |
| 25° | 0.423 | 2.366 | 0.466 | 2.145 | 1.103 | 0.906 | 65° |
| 26 | 0.438 | 2.281 | 0.488 | 2.050 | 1.113 | 0.899 | 64 |
| 27 | 0.454 | 2.203 | 0.510 | 1.963 | 1.122 | 0.891 | 63 |
| 28 | 0.469 | 2.130 | 0.532 | 1.881 | 1.133 | 0.883 | 62 |
| 29 | 0.485 | 2.063 | 0.554 | 1.804 | 1.143 | 0.875 | 61 |
| 30° | 0.500 | 2.000 | 0.577 | 1.732 | 1.155 | 0.866 | 60° |
| 31 | 0.515 | 1.942 | 0.601 | 1.664 | 1.167 | 0.857 | 59 |
| 32 | 0.530 | 1.887 | 0.625 | 1.600 | 1.179 | 0.848 | 58 |
| 33 | 0.545 | 1.836 | 0.649 | 1.540 | 1.192 | 0.839 | 57 |
| 34 | 0.559 | 1.788 | 0.675 | 1.483 | 1.206 | 0.829 | 56 |
| 35° | 0.574 | 1.743 | 0.700 | 1.428 | 1.221 | 0.819 | 55° |
| 36 | 0.588 | 1.701 | 0.727 | 1.376 | 1.236 | 0.809 | 54 |
| 37 | 0.602 | 1.662 | 0.754 | 1.327 | 1.252 | 0.799 | 53 |
| 38 | 0.616 | 1.624 | 0.781 | 1.280 | 1.269 | 0.788 | 52 |
| 39 | 0.629 | 1.589 | 0.810 | 1.235 | 1.287 | 0.777 | 51 |
| 40° | 0.643 | 1.556 | 0.839 | 1.192 | 1.305 | 0.766 | 50° |
| 41 | 0.656 | 1.524 | 0.869 | 1.150 | 1.325 | 0.755 | 49 |
| 42 | 0.669 | 1.494 | 0.900 | 1.111 | 1.346 | 0.743 | 48 |
| 43 | 0.682 | 1.466 | 0.933 | 1.072 | 1.367 | 0.731 | 47 |
| 44 | 0.695 | 1.440 | 0.966 | 1.036 | 1.390 | 0.719 | 46 |
| 45° | 0.707 | 1.414 | 1.000 | 1.000 | 1.414 | 0.707 | 45° |
| | Cos. | Sec. | Ctn. | Tan. | Csc. | Sin. | Angle. |

## Logarithms.

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | P. P. 1. 2. 3. 4. 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **10** | 0000 | 0043 | 0086 | 0128 | 0170 | 0212 | 0253 | 0294 | 0334 | 0374 | 4. 8.12.17.21 |
| 11 | 0414 | 0453 | 0492 | 0531 | 0569 | 0607 | 0645 | 0682 | 0719 | 0755 | 4 8.11.15.19 |
| 12 | 0792 | 0828 | 0864 | 0899 | 0934 | 0969 | 1004 | 1038 | 1072 | 1106 | 3. 7.10.14.17 |
| 13 | 1139 | 1173 | 1206 | 1239 | 1271 | 1303 | 1335 | 1367 | 1399 | 1430 | 3. 6.10.13.16 |
| 14 | 1461 | 1492 | 1523 | 1553 | 1584 | 1614 | 1644 | 1673 | 1703 | 1732 | 3. 6. 9.12.15 |
| **15** | 1761 | 1790 | 1818 | 1847 | 1875 | 1903 | 1931 | 1959 | 1987 | 2014 | 3. 6. 8.11.14 |
| 16 | 2041 | 2068 | 2095 | 2122 | 2148 | 2175 | 2201 | 2227 | 2253 | 2279 | 3. 5. 8.11.13 |
| 17 | 2304 | 2330 | 2355 | 2380 | 2405 | 2430 | 2455 | 2480 | 2504 | 2529 | 2. 5. 7.10.12 |
| 18 | 2553 | 2577 | 2601 | 2625 | 2648 | 2672 | 2695 | 2718 | 2742 | 2765 | 2. 5. 7. 9.12 |
| 19 | 2788 | 2810 | 2833 | 2856 | 2878 | 2900 | 2923 | 2945 | 2967 | 2989 | 2. 4. 7. 9.11 |
| **20** | 3010 | 3032 | 3054 | 3075 | 3096 | 3118 | 3139 | 3160 | 3181 | 3201 | 2. 4. 6. 8.11 |
| 21 | 3222 | 3243 | 3263 | 3284 | 3304 | 3324 | 3345 | 3365 | 3385 | 3404 | 2. 4. 6. 8.10 |
| 22 | 3424 | 3444 | 3464 | 3483 | 3502 | 3522 | 3541 | 3560 | 3579 | 3598 | 2. 4. 6. 8.10 |
| 23 | 3617 | 3636 | 3655 | 3674 | 3692 | 3711 | 3729 | 3747 | 3766 | 3784 | 2. 4. 5. 7. 9 |
| 24 | 3802 | 3820 | 3838 | 3856 | 3874 | 3892 | 3909 | 3927 | 3945 | 3962 | 2. 4. 5. 7. 9 |
| **25** | 3979 | 3997 | 4014 | 4031 | 4048 | 4065 | 4082 | 4099 | 4116 | 4133 | 2 3. 5. 7. 9 |
| 26 | 4150 | 4166 | 4183 | 4200 | 4216 | 4232 | 4249 | 4265 | 4281 | 4298 | 2. 3. 5. 7. 8 |
| 27 | 4314 | 4330 | 4346 | 4362 | 4378 | 4393 | 4409 | 4425 | 4440 | 4456 | 2. 3. 5. 6. 8 |
| 28 | 4472 | 4487 | 4502 | 4518 | 4533 | 4548 | 4564 | 4579 | 4594 | 4609 | 2. 3. 5. 6. 8 |
| 29 | 4624 | 4639 | 4654 | 4669 | 4683 | 4698 | 4713 | 4728 | 4742 | 4757 | 1. 3. 4. 6. 7 |
| **30** | 4771 | 4786 | 4800 | 4814 | 4829 | 4843 | 4857 | 4871 | 4886 | 4900 | 1. 3. 4. 6. 7 |
| 31 | 4914 | 4928 | 4942 | 4955 | 4969 | 4983 | 4997 | 5011 | 5024 | 5038 | 1. 3. 4. 6. 7 |
| 32 | 5051 | 5065 | 5079 | 5092 | 5105 | 5119 | 5132 | 5145 | 5159 | 5172 | 1. 3. 4. 5. 7 |
| 33 | 5185 | 5198 | 5211 | 5224 | 5237 | 5250 | 5263 | 5276 | 5289 | 5302 | 1. 3. 4. 5. 6 |
| 34 | 5315 | 5328 | 5340 | 5353 | 5366 | 5378 | 5391 | 5403 | 5416 | 5428 | 1. 3. 4. 5. 6 |
| **35** | 5441 | 5453 | 5465 | 5478 | 5490 | 5502 | 5514 | 5527 | 5539 | 5551 | 1. 2. 4. 5. 6 |
| 36 | 5563 | 5575 | 5587 | 5599 | 5611 | 5623 | 5635 | 5647 | 5658 | 5670 | 1. 2. 4. 5. 6 |
| 37 | 5682 | 5694 | 5705 | 5717 | 5729 | 5740 | 5752 | 5763 | 5775 | 5786 | 1 2. 3. 5. 6 |
| 38 | 5798 | 5809 | 5821 | 5832 | 58.3 | 5855 | 5866 | 5877 | 5888 | 5899 | 1. 2. 3. 5. 6 |
| 39 | 5911 | 5922 | 5933 | 5944 | 5955 | 5966 | 5977 | 5988 | 5999 | 6010 | 1. 2. 3. 4. 6 |
| **40** | 6021 | 6031 | 6042 | 6053 | 6064 | 6075 | 6085 | 6096 | 6107 | 6117 | 1. 2. 3. 4. 5 |
| 41 | 6128 | 6138 | 6149 | 6160 | 6170 | 6180 | 6191 | 6201 | 6212 | 6222 | 1. 2. 3. 4. 5 |
| 42 | 6232 | 6243 | 6253 | 6263 | 6274 | 6284 | 6294 | 6304 | 6314 | 6325 | 1. 2. 3. 4. 5 |
| 43 | 6335 | 6345 | 6355 | 6365 | 6375 | 6385 | 6395 | 6405 | 6415 | 6425 | 1. 2. 3. 4. 5 |
| 44 | 6435 | 6444 | 6454 | 6464 | 6474 | 6484 | 6493 | 6503 | 6513 | 6522 | 1. 2. 3. 4. 5 |
| **45** | 6532 | 6542 | 6551 | 6561 | 6571 | 6580 | 6590 | 6599 | 6609 | 6618 | 1. 2. 3. 4. 5 |
| 46 | 6628 | 6637 | 6646 | 6656 | 6665 | 6675 | 6684 | 6693 | 6702 | 6' .2 | 1 2. 3. 4. 5 |
| 47 | 6721 | 6730 | 6739 | 6749 | 6758 | 6767 | 6776 | 6785 | 6794 | 6803 | 1. 2. 3. 4. 5 |
| 48 | 6812 | 6821 | 6830 | 6839 | 6848 | 6857 | 6866 | 6875 | 6884 | 6893 | 1. 2. 3 4. 4 |
| 49 | 6902 | 6911 | 6920 | 6928 | 6937 | 6946 | 6955 | 6964 | 6972 | 6981 | 1. 2. 3. 4. 4 |
| **50** | 6990 | 6998 | 7007 | 7016 | 7024 | 7033 | 7042 | 7050 | 7059 | 7067 | 1. 2. 3. 3. 4 |
| 51 | 7076 | 7084 | 7093 | 7101 | 7110 | 7118 | 7126 | 7135 | 7143 | 7152 | 1. 2. 3. 3. 4 |
| 52 | 7160 | 7168 | 7177 | 7185 | 7193 | 7202 | 7210 | 7218 | 7226 | 7235 | 1. 2 2. 3 4 |
| 53 | 7243 | 7251 | 7259 | 7267 | 7275 | 7284 | 7292 | 7300 | 7308 | 7316 | 1. 2. 2 3. 4 |
| 54 | 7324 | 7332 | 7340 | 7348 | 7356 | 7364 | 7372 | 7380 | 7388 | 7396 | 1. 2. 2. 3. 4 |

NOTE.—This page and the three that follow it are taken from the Mathematical Tables of Prof. J. M. Peirce, published by Messrs. Ginn & Co.

## Logarithms.

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | P. P. 1. 2. 3. 4. 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **55** | 7404 | 7412 | 7419 | 7427 | 7435 | 7443 | 7451 | 7459 | 7466 | 7474 | 1. 2. 2. 3. 4 |
| 56 | 7482 | 7490 | 7497 | 7505 | 7513 | 7520 | 7528 | 7536 | 7543 | 7551 | 1. 2. 2. 3. 4 |
| 57 | 7559 | 7566 | 7574 | 7582 | 7589 | 7597 | 7604 | 7612 | 7619 | 7627 | 1. 2. 2. 3. 4 |
| 58 | 7634 | 7642 | 7649 | 7657 | 7664 | 7672 | 7679 | 7686 | 7694 | 7701 | 1. 1. 2. 3. 4 |
| 59 | 7709 | 7716 | 7723 | 7731 | 7738 | 7745 | 7752 | 7760 | 7767 | 7774 | 1. 1. 2. 3. 4 |
| **60** | 7782 | 7789 | 7796 | 7803 | 7810 | 7818 | 7825 | 7832 | 7839 | 7846 | 1. 1. 2. 3. 4 |
| 61 | 7853 | 7860 | 7868 | 7875 | 7882 | 7889 | 7896 | 7903 | 7910 | 7917 | 1. 1. 2. 3. 4 |
| 62 | 7924 | 7931 | 7938 | 7945 | 7952 | 7959 | 7966 | 7973 | 7980 | 7987 | 1. 1. 2. 3. 3 |
| 63 | 7993 | 8000 | 8007 | 8014 | 8021 | 8028 | 8035 | 8041 | 8048 | 8055 | 1. 1. 2. 3. 3 |
| 64 | 8062 | 8069 | 8075 | 8082 | 8089 | 8096 | 8102 | 8109 | 8116 | 8122 | 1. 1. 2. 3. 3 |
| **65** | 8129 | 8136 | 8142 | 8149 | 8156 | 8162 | 8169 | 8176 | 8182 | 8189 | 1. 1. 2. 3. 3 |
| 66 | 8195 | 8202 | 8209 | 8215 | 8222 | 8228 | 8235 | 8241 | 8248 | 8254 | 1. 1. 2. 3. 3 |
| 67 | 8261 | 8267 | 8274 | 8280 | 8287 | 8293 | 8299 | 8306 | 8312 | 8319 | 1. 1. 2. 3. 3 |
| 68 | 8325 | 8331 | 8338 | 8344 | 8351 | 8357 | 8363 | 8370 | 8376 | 8382 | 1. 1. 2. 3. 3 |
| 69 | 8388 | 8395 | 8401 | 8407 | 8414 | 8420 | 8426 | 8432 | 8439 | 8445 | 1. 1. 2. 3. 3 |
| **70** | 8451 | 8457 | 8463 | 8470 | 8476 | 8482 | 8488 | 8494 | 8500 | 8506 | 1. 1. 2. 2. 3 |
| 71 | 8513 | 8519 | 8525 | 8531 | 8537 | 8543 | 8549 | 8555 | 8561 | 8567 | 1. 1. 2. 2. 3 |
| 72 | 8573 | 8579 | 8585 | 8591 | 8597 | 8603 | 8609 | 8615 | 8621 | 8627 | 1. 1. 2. 2. 3 |
| 73 | 8633 | 8639 | 8645 | 8651 | 8657 | 8663 | 8669 | 8675 | 8681 | 8686 | 1. 1. 2. 2. 3 |
| 74 | 8692 | 8698 | 8704 | 8710 | 8716 | 8722 | 8727 | 8733 | 8739 | 8745 | 1. 1. 2. 2. 3 |
| **75** | 8751 | 8756 | 8762 | 8768 | 8774 | 8779 | 8785 | 8791 | 8797 | 8802 | 1. 1. 2. 2. 3 |
| 76 | 8808 | 8814 | 8820 | 8825 | 8831 | 8837 | 8842 | 8848 | 8854 | 8859 | 1. 1. 2. 2. 3 |
| 77 | 8865 | 8871 | 8876 | 8882 | 8887 | 8893 | 8899 | 8904 | 8910 | 8915 | 1. 1. 2. 2. 3 |
| 78 | 8921 | 8927 | 8932 | 8938 | 8943 | 8949 | 8954 | 8960 | 8965 | 8971 | 1. 1. 2. 2. 3 |
| 79 | 8976 | 8982 | 8987 | 8993 | 8998 | 9004 | 9009 | 9015 | 9020 | 9025 | 1. 1. 2. 2. 3 |
| **80** | 9031 | 9036 | 9042 | 9047 | 9053 | 9058 | 9063 | 9069 | 9074 | 9079 | 1. 1. 2. 2. 3 |
| 81 | 9085 | 9090 | 9096 | 9101 | 9106 | 9112 | 9117 | 9122 | 9128 | 9133 | 1. 1. 2. 2. 3 |
| 82 | 9138 | 9143 | 9149 | 9154 | 9159 | 9165 | 9170 | 9175 | 9180 | 9186 | 1. 1. 2. 2 3 |
| 83 | 9191 | 9196 | 9201 | 9206 | 9212 | 9217 | 9222 | 9227 | 9232 | 9238 | 1. 1. 2. 2. 3 |
| 84 | 9243 | 9248 | 9253 | 9258 | 9263 | 9269 | 9274 | 9279 | 9284 | 9289 | 1. 1. 2. 2. 3 |
| **85** | 9294 | 9299 | 9304 | 9309 | 9315 | 9320 | 9325 | 9330 | 9335 | 9340 | 1. 1. 2 2. 3 |
| 86 | 9345 | 9350 | 9355 | 9360 | 9365 | 9370 | 9375 | 9380 | 9385 | 9390 | 1. 1. 2. 2. 3 |
| 87 | 9395 | 9400 | 9405 | 9410 | 9415 | 9420 | 9425 | 9430 | 9435 | 9440 | 0. 1. 1. 2. 2 |
| 88 | 9445 | 9450 | 9455 | 9460 | 9465 | 9469 | 9474 | 9479 | 9484 | 9489 | 0. 1. 1. 2. 2 |
| 89 | 9494 | 9499 | 9504 | 9509 | 9513 | 9518 | 9523 | 9528 | 9533 | 9538 | 0. 1. 1. 2. 2 |
| **90** | 9542 | 9547 | 9552 | 9557 | 9562 | 9566 | 9571 | 9576 | 9581 | 9586 | 0. 1. 1. 2. 2 |
| 91 | 9590 | 9595 | 9600 | 9605 | 9609 | 9614 | 9619 | 9624 | 9628 | 9633 | 0. 1. 1. 2. 2 |
| 92 | 9638 | 9643 | 9647 | 9652 | 9657 | 9661 | 9666 | 9671 | 9675 | 9680 | 0 1. 1. 2. 2 |
| 93 | 9685 | 9689 | 9694 | 9699 | 9703 | 9708 | 9713 | 9717 | 9722 | 9727 | 0. 1. 1. 2. 2 |
| 94 | 9731 | 9736 | 9741 | 9745 | 9750 | 9754 | 9759 | 9763 | 9768 | 9773 | 0 1. 1. 2 2 |
| **95** | 9777 | 9782 | 9786 | 9791 | 9795 | 9800 | 9805 | 9809 | 9814 | 9818 | 0. 1 1. 2. 2 |
| 96 | 9823 | 9827 | 9832 | 9836 | 9841 | 9845 | 9850 | 9854 | 9859 | 9863 | 0. 1. 1. 2. 2 |
| 97 | 9868 | 9872 | 9877 | 9881 | 9886 | 9890 | 9894 | 9899 | 9903 | 9908 | 0. 1. 1. 2. 2 |
| 98 | 9912 | 9917 | 9921 | 9926 | 9930 | 9934 | 9939 | 9943 | 9948 | 9952 | 0 1. 1. 2. 2 |
| 99 | 9956 | 9961 | 9965 | 9969 | 9974 | 9978 | 9983 | 9987 | 9991 | 9996 | 0. 1. 1. 2 2 |

$\log \pi = 0.49715 -$. $\log e = 0.43429 +$.

## Logarithms.

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **100** | 0000 | 0004 | 0009 | 0013 | 0017 | 0022 | 0026 | 0030 | 0035 | 0039 | 0043 |
| 101 | 0043 | 0048 | 0052 | 0056 | 0060 | 0065 | 0069 | 0073 | 0077 | 0082 | 0086 |
| 102 | 0086 | 0090 | 0095 | 0099 | 0103 | 0107 | 0111 | 0116 | 0120 | 0124 | 0128 |
| 103 | 0128 | 0133 | 0137 | 0141 | 0145 | 0149 | 0154 | 0158 | 0162 | 0166 | 0170 |
| 104 | 0170 | 0175 | 0179 | 0183 | 0187 | 0191 | 0195 | 0199 | 0204 | 0208 | 0212 |
| **105** | 0212 | 0216 | 0220 | 0224 | 0228 | 0233 | 0237 | 0241 | 0245 | 0249 | 0253 |
| 106 | 0253 | 0257 | 0261 | 0265 | 0269 | 0273 | 0278 | 0282 | 0286 | 0290 | 0294 |
| 107 | 0294 | 0298 | 0302 | 0306 | 0310 | 0314 | 0318 | 0322 | 0326 | 0330 | 0334 |
| 108 | 0334 | 0338 | 0342 | 0346 | 0350 | 0354 | 0358 | 0362 | 0366 | 0370 | 0374 |
| 109 | 0374 | 0378 | 0382 | 0386 | 0390 | 0394 | 0398 | 0402 | 0406 | 0410 | 0414 |
| **110** | 0414 | 0418 | 0422 | 0426 | 0430 | 0434 | 0438 | 0441 | 0445 | 0449 | 0453 |
| 111 | 0453 | 0457 | 0461 | 0465 | 0469 | 0473 | 0477 | 0481 | 0484 | 0488 | 0492 |
| 112 | 0492 | 0496 | 0500 | 0504 | 0508 | 0512 | 0515 | 0519 | 0523 | 0527 | 0531 |
| 113 | 0531 | 0535 | 0538 | 0542 | 0546 | 0550 | 0554 | 0558 | 0561 | 0565 | 0569 |
| 114 | 0569 | 0573 | 0577 | 0580 | 0584 | 0588 | 0592 | 0596 | 0599 | 0603 | 0607 |
| **115** | 0607 | 0611 | 0615 | 0618 | 0622 | 0626 | 0630 | 0633 | 0637 | 0641 | 0645 |
| 116 | 0645 | 0648 | 0652 | 0656 | 0660 | 0663 | 0667 | 0671 | 0674 | 0678 | 0682 |
| 117 | 0682 | 0686 | 0689 | 0693 | 0697 | 0700 | 0704 | 0708 | 0711 | 0715 | 0719 |
| 118 | 0719 | 0722 | 0726 | 0730 | 0734 | 0737 | 0741 | 0745 | 0748 | 0752 | 0755 |
| 119 | 0755 | 0759 | 0763 | 0766 | 0770 | 0774 | 0777 | 0781 | 0785 | 0788 | 0792 |
| **120** | 0792 | 0795 | 0799 | 0803 | 0806 | 0810 | 0813 | 0817 | 0821 | 0824 | 0828 |
| 121 | 0828 | 0831 | 0835 | 0839 | 0842 | 0846 | 0849 | 0853 | 0856 | 0860 | 0864 |
| 122 | 0864 | 0867 | 0871 | 0874 | 0878 | 0881 | 0885 | 0888 | 0892 | 0896 | 0899 |
| 123 | 0899 | 0903 | 0906 | 0910 | 0913 | 0917 | 0920 | 0924 | 0927 | 0931 | 0934 |
| 124 | 0934 | 0938 | 0941 | 0945 | 0948 | 0952 | 0955 | 0959 | 0962 | 0966 | 0969 |
| **125** | 0969 | 0973 | 0976 | 0980 | 0983 | 0986 | 0990 | 0993 | 0997 | 1000 | 1004 |
| 126 | 1004 | 1007 | 1011 | 1014 | 1017 | 1021 | 1024 | 1028 | 1031 | 1035 | 1038 |
| 127 | 1038 | 1041 | 1045 | 1048 | 1052 | 1055 | 1059 | 1062 | 1065 | 1069 | 1072 |
| 128 | 1072 | 1075 | 1079 | 1082 | 1086 | 1089 | 1092 | 1096 | 1099 | 1103 | 1106 |
| 129 | 1106 | 1109 | 1113 | 1116 | 1119 | 1123 | 1126 | 1129 | 1133 | 1136 | 1139 |
| **130** | 1139 | 1143 | 1146 | 1149 | 1153 | 1156 | 1159 | 1163 | 1166 | 1169 | 1173 |
| 131 | 1173 | 1176 | 1179 | 1183 | 1186 | 1189 | 1193 | 1196 | 1199 | 1202 | 1206 |
| 132 | 1206 | 1209 | 1212 | 1216 | 1219 | 1222 | 1225 | 1229 | 1232 | 1235 | 1239 |
| 133 | 1239 | 1242 | 1245 | 1248 | 1252 | 1255 | 1258 | 1261 | 1265 | 1268 | 1271 |
| 134 | 1271 | 1274 | 1278 | 1281 | 1284 | 1287 | 1290 | 1294 | 1297 | 1300 | 1303 |
| **135** | 1303 | 1307 | 1310 | 1313 | 1316 | 1319 | 1323 | 1326 | 1329 | 1332 | 1335 |
| 136 | 1335 | 1339 | 1342 | 1345 | 1348 | 1351 | 1355 | 1358 | 1361 | 1364 | 1367 |
| 137 | 1367 | 1370 | 1374 | 1377 | 1380 | 1383 | 1386 | 1389 | 1392 | 1396 | 1399 |
| 138 | 1399 | 1402 | 1405 | 1408 | 1411 | 1414 | 1418 | 1421 | 1424 | 1427 | 1430 |
| 139 | 1430 | 1433 | 1436 | 1440 | 1443 | 1446 | 1449 | 1452 | 1455 | 1458 | 1461 |
| **140** | 1461 | 1464 | 1467 | 1471 | 1474 | 1477 | 1480 | 1483 | 1486 | 1489 | 1492 |
| 141 | 1492 | 1495 | 1498 | 1501 | 1504 | 1508 | 1511 | 1514 | 1517 | 1520 | 1523 |
| 142 | 1523 | 1526 | 1529 | 1532 | 1535 | 1538 | 1541 | 1544 | 1547 | 1550 | 1553 |
| 143 | 1553 | 1556 | 1559 | 1562 | 1565 | 1569 | 1572 | 1575 | 1578 | 1581 | 1584 |
| 144 | 1584 | 1587 | 1590 | 1593 | 1596 | 1599 | 1602 | 1605 | 1608 | 1611 | 1614 |
| **145** | 1614 | 1617 | 1620 | 1623 | 1626 | 1629 | 1632 | 1635 | 1638 | 1641 | 1644 |
| 146 | 1644 | 1647 | 1649 | 1652 | 1655 | 1658 | 1661 | 1664 | 1367 | 1670 | 1673 |
| 147 | 1673 | 1676 | 1679 | 1682 | 1685 | 1688 | 1691 | 1694 | 1697 | 1700 | 1703 |
| 148 | 1703 | 1706 | 1708 | 1711 | 1714 | 1717 | 1720 | 1723 | 1726 | 1729 | 1732 |
| 149 | 1732 | 1735 | 1738 | 1741 | 1744 | 1746 | 1749 | 1752 | 1755 | 1758 | 1761 |

## Logarithms.

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **150** | 1761 | 1764 | 1767 | 1770 | 1772 | 1775 | 1778 | 1781 | 1784 | 1787 | 1790 |
| 151 | 1790 | 1793 | 1796 | 1798 | 1801 | 1804 | 1807 | 1810 | 1813 | 1816 | 1818 |
| 152 | 1818 | 1821 | 1824 | 1827 | 1830 | 1833 | 1836 | 1838 | 1841 | 1844 | 1847 |
| 153 | 1847 | 1850 | 1853 | 1855 | 1858 | 1861 | 1864 | 1867 | 1870 | 1872 | 1875 |
| 154 | 1875 | 1878 | 1881 | 1884 | 1886 | 1889 | 1892 | 1895 | 1898 | 1901 | 1903 |
| **155** | 1903 | 1906 | 1909 | 1912 | 1915 | 1917 | 1920 | 1923 | 1926 | 1928 | 1931 |
| 156 | 1931 | 1934 | 1937 | 1940 | 1942 | 1945 | 1948 | 1951 | 1953 | 1956 | 1959 |
| 157 | 1959 | 1962 | 1965 | 1967 | 1970 | 1973 | 1976 | 1978 | 1981 | 1984 | 1987 |
| 158 | 1987 | 1989 | 1992 | 1995 | 1998 | 2000 | 2003 | 2006 | 2009 | 2011 | 2014 |
| 159 | 2014 | 2017 | 2019 | 2022 | 2025 | 2028 | 2030 | 2033 | 2036 | 2038 | 2041 |
| **160** | 2041 | 2044 | 2047 | 2049 | 2052 | 2055 | 2057 | 2060 | 2063 | 2066 | 2068 |
| 161 | 2068 | 2071 | 2074 | 2076 | 2079 | 2082 | 2084 | 2087 | 2090 | 2092 | 2095 |
| 162 | 2095 | 2098 | 2101 | 2103 | 2106 | 2109 | 2111 | 2114 | 2117 | 2119 | 2122 |
| 163 | 2122 | 2125 | 2127 | 2130 | 2133 | 2135 | 2138 | 2140 | 2143 | 2146 | 2148 |
| 164 | 2148 | 2151 | 2154 | 2156 | 2159 | 2162 | 2164 | 2167 | 2170 | 2172 | 2175 |
| **165** | 2175 | 2177 | 2180 | 2183 | 2185 | 2188 | 2191 | 2193 | 2196 | 2198 | 2201 |
| 166 | 2201 | 2204 | 2206 | 2209 | 2212 | 2214 | 2217 | 2219 | 2222 | 2225 | 2227 |
| 167 | 2227 | 2230 | 2232 | 2235 | 2238 | 2240 | 2243 | 2245 | 2248 | 2251 | 2253 |
| 168 | 2253 | 2256 | 2258 | 2261 | 2263 | 2266 | 2269 | 2271 | 2274 | 2276 | 2279 |
| 169 | 2279 | 2281 | 2284 | 2287 | 2289 | 2292 | 2294 | 2297 | 2299 | 2302 | 2304 |
| **170** | 2304 | 2307 | 2310 | 2312 | 2315 | 2317 | 2320 | 2322 | 2325 | 2327 | 2330 |
| 171 | 2330 | 2333 | 2335 | 2338 | 2340 | 2343 | 2345 | 2348 | 2350 | 2353 | 2355 |
| 172 | 2355 | 2358 | 2360 | 2363 | 2365 | 2368 | 2370 | 2373 | 2375 | 2378 | 2380 |
| 173 | 2380 | 2383 | 2385 | 2388 | 2390 | 2393 | 2395 | 2398 | 2400 | 2403 | 2405 |
| 174 | 2405 | 2408 | 2410 | 2413 | 2415 | 2418 | 2420 | 2423 | 2425 | 2428 | 2430 |
| **175** | 2430 | 2433 | 2435 | 2438 | 2440 | 2443 | 2445 | 2448 | 2450 | 2453 | 2455 |
| 176 | 2455 | 2458 | 2460 | 2463 | 2465 | 2467 | 2470 | 2472 | 2475 | 2477 | 2480 |
| 177 | 2480 | 2482 | 2485 | 2487 | 2490 | 2492 | 2494 | 2497 | 2499 | 2502 | 2504 |
| 178 | 2504 | 2507 | 2509 | 2512 | 2514 | 2516 | 2519 | 2521 | 2524 | 2526 | 2529 |
| 179 | 2529 | 2531 | 2533 | 2536 | 2538 | 2541 | 2543 | 2545 | 2548 | 2550 | 2553 |
| **180** | 2553 | 2555 | 2558 | 2560 | 2562 | 2565 | 2567 | 2570 | 2572 | 2574 | 2577 |
| 181 | 2577 | 2579 | 2582 | 2584 | 2586 | 2589 | 2591 | 2594 | 2596 | 2598 | 2601 |
| 182 | 2601 | 2603 | 2605 | 2608 | 2610 | 2613 | 2615 | 2617 | 2620 | 2622 | 2625 |
| 183 | 2625 | 2627 | 2629 | 2632 | 2634 | 2636 | 2639 | 2641 | 2643 | 2646 | 2648 |
| 184 | 2648 | 2651 | 2653 | 2655 | 2658 | 2660 | 2662 | 2665 | 2667 | 2669 | 2672 |
| **185** | 2672 | 2674 | 2676 | 2679 | 2681 | 2683 | 2686 | 2688 | 2690 | 2693 | 2695 |
| 186 | 2695 | 2697 | 2700 | 2702 | 2704 | 2707 | 2709 | 2711 | 2714 | 2716 | 2718 |
| 187 | 2718 | 2721 | 2723 | 2725 | 2728 | 2730 | 2732 | 2735 | 2737 | 2739 | 2742 |
| 188 | 2742 | 2744 | 2746 | 2749 | 2751 | 2753 | 2755 | 2758 | 2760 | 2762 | 2765 |
| 189 | 2765 | 2767 | 2769 | 2772 | 2774 | 2776 | 2778 | 2781 | 2783 | 2785 | 2788 |
| **190** | 2788 | 2790 | 2792 | 2794 | 2797 | 2799 | 2801 | 2804 | 2806 | 2808 | 2810 |
| 191 | 2810 | 2813 | 2815 | 2817 | 2819 | 2822 | 2824 | 2826 | 2828 | 2831 | 2833 |
| 192 | 2833 | 2835 | 2838 | 2840 | 2842 | 2844 | 2847 | 2849 | 2851 | 2853 | 2856 |
| 193 | 2856 | 2858 | 2860 | 2862 | 2865 | 2867 | 2869 | 2871 | 2874 | 2876 | 2878 |
| 194 | 2878 | 2880 | 2882 | 2885 | 2887 | 2889 | 2891 | 2894 | 2896 | 2898 | 2900 |
| **195** | 2900 | 2903 | 2905 | 2907 | 2909 | 2911 | 2914 | 2916 | 2918 | 2920 | 2923 |
| 196 | 2923 | 2925 | 2927 | 2929 | 2931 | 2934 | 2936 | 2938 | 2940 | 2942 | 2945 |
| 197 | 2945 | 2947 | 2949 | 2951 | 2953 | 2956 | 2958 | 2960 | 2962 | 2964 | 2967 |
| 198 | 2967 | 2969 | 2971 | 2973 | 2975 | 2978 | 2980 | 2982 | 2984 | 2986 | 2989 |
| 199 | 2989 | 2991 | 2993 | 2995 | 2997 | 2999 | 3002 | 3004 | 3006 | 3008 | 3010 |

## Trigonometric Functions.

| RADIANS. | DEGREES. | SINES. Nat. | SINES. Log. | COSINES. Nat. | COSINES. Log. | TANGENTS. Nat. | TANGENTS. Log. | COTANGENTS. Nat. | COTANGENTS. Log. | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0000 | 0° 00′ | .0000 | ∞ | 1.0000 | 0.0000 | .0000 | ∞ | ∞ | ∞ | 90° 00′ | 1.5708 |
| 0.0029 | 10 | .0029 | 7.4637 | 1.0000 | .0000 | .0029 | 7.4637 | 343.77 | 2.5363 | 50 | 1.5679 |
| 0.0058 | 20 | .0058 | .7648 | 1.0000 | .0000 | .0058 | .7648 | 171.89 | .2352 | 40 | 1.5650 |
| 0.0087 | 30 | .0087 | .9408 | 1.0000 | .0000 | .0087 | .9409 | 114.59 | .0591 | 30 | 1.5621 |
| 0.0116 | 40 | .0116 | 8.0658 | .9999 | .0000 | .0116 | 8.0658 | 85.940 | 1.9342 | 20 | 1.5592 |
| 0.0145 | 50 | .0145 | .1627 | .9999 | .0000 | .0145 | .1627 | 68.750 | .8373 | 10 | 1.5563 |
| 0.0175 | 1° 00′ | .0175 | 8.2419 | .9998 | 9.9999 | .0175 | 8.2419 | 57.290 | 1.7581 | 89° 00′ | 1.5533 |
| 0.0204 | 10 | .0204 | .3088 | .9998 | .9999 | .0204 | .3089 | 49.104 | .6911 | 50 | 1.5504 |
| 0.0233 | 20 | .0233 | .3668 | .9997 | .9999 | .0233 | .3669 | 42.964 | .6331 | 40 | 1.5475 |
| 0.0262 | 30 | .0262 | .4179 | .9997 | .9999 | .0262 | .4181 | 38.188 | .5819 | 30 | 1.5446 |
| 0.0291 | 40 | .0291 | .4637 | .9996 | .9998 | .0291 | .4638 | 34.368 | .5362 | 20 | 1.5417 |
| 0.0320 | 50 | .0320 | .5050 | .9995 | .9998 | .0320 | .5053 | 31.242 | .4947 | 10 | 1.5388 |
| 0.0349 | 2° 00′ | .0349 | 8.5428 | .9994 | 9.9997 | .0349 | 8.5431 | 28 636 | 1.4569 | 88° 00′ | 1.5359 |
| 0.0378 | 10 | .0378 | .5776 | .9993 | .9997 | .0378 | .5779 | 26.432 | .4221 | 50 | 1.5330 |
| 0.0407 | 20 | .0407 | .6097 | .9992 | .9996 | .0407 | .6101 | 24.542 | .3899 | 40 | 1.5301 |
| 0.0436 | 30 | .0436 | .6397 | .9990 | .9996 | .0437 | .6401 | 22.904 | .3599 | 30 | 1.5272 |
| 0.0465 | 40 | .0465 | .6677 | .9989 | .9995 | .0466 | .6682 | 21.470 | .3318 | 20 | 1.5243 |
| 0.0495 | 50 | .0494 | .6940 | .9988 | .9995 | .0495 | .6945 | 20.206 | .3055 | 10 | 1.5213 |
| 0.0524 | 3° 00′ | .0523 | 8.7188 | .9986 | 9.9994 | .0524 | 8.7194 | 19.081 | 1.2806 | 87° 00′ | 1.5184 |
| 0.0553 | 10 | .0552 | .7423 | .9985 | .9993 | .0553 | .7429 | 18 075 | .2571 | 50 | 1.5155 |
| 0.0582 | 20 | .0581 | .7645 | .9983 | .9993 | .0582 | .7652 | 17.169 | .2348 | 40 | 1.5126 |
| 0.0611 | 30 | .0610 | .7857 | .9981 | .9992 | .0612 | .7865 | 16.350 | .2135 | 30 | 1.5097 |
| 0.0640 | 40 | .0640 | .8059 | .9980 | .9991 | .0641 | .8067 | 15 605 | .1933 | 20 | 1.5068 |
| 0.0669 | 50 | .0669 | .8251 | .9978 | .9990 | .0670 | .8261 | 14.924 | .1739 | 10 | 1.5039 |
| 0.0698 | 4° 00′ | .0698 | 8.8436 | .9976 | 9.9989 | .0699 | 8.8446 | 14.301 | 1.1554 | 86° 00′ | 1 5010 |
| 0.0727 | 10 | .0727 | .8613 | .9974 | .9989 | .0729 | .8624 | 13.727 | .1376 | 50 | 1 4981 |
| 0.0756 | 20 | .0756 | .8783 | .9971 | .9988 | .0758 | .8795 | 13.197 | .1205 | 40 | 1.4952 |
| 0.0785 | 30 | .0785 | .8946 | .9969 | .9987 | .0787 | .8960 | 12 706 | .1040 | 30 | 1.4923 |
| 0.0814 | 40 | .0814 | .9104 | .9967 | .9986 | .0816 | .9118 | 12.251 | .0882 | 20 | 1.4893 |
| 0.0844 | 50 | .0843 | .9256 | .9964 | .9985 | .0846 | .9272 | 11.826 | .0728 | 10 | 1.4864 |
| 0.0873 | 5° 00′ | .0872 | 8.9403 | .9962 | 9.9983 | .0875 | 8.9420 | 11.430 | 1.0580 | 85° 00′ | 1.4835 |
| 0.0902 | 10 | 0901 | .9545 | .9959 | .9982 | .0904 | .9563 | 11.059 | .0437 | 50 | 1.4806 |
| 0.0931 | 20 | .0929 | .9682 | .9957 | .9981 | .0934 | .9701 | 10 712 | .0299 | 40 | 1.4777 |
| 0.0960 | 30 | .0958 | .9816 | .9954 | .9980 | .0963 | .9836 | 10.385 | .0164 | 30 | 1.4748 |
| 0.0989 | 40 | .0987 | .9945 | .9951 | .9979 | .0992 | .9966 | 10.078 | .0034 | 20 | 1.4719 |
| 0.1018 | 50 | .1016 | 9.0070 | .9948 | .9977 | .1022 | 9.0093 | 9.7882 | 0.9907 | 10 | 1.4690 |
| 0.1047 | 6° 00′ | .1045 | 9.0192 | .9945 | 9.9976 | .1051 | 9.0216 | 9.5144 | 0.9784 | 84° 00′ | 1.4661 |
| 0.1076 | 10 | 1074 | .0311 | .9942 | .9975 | .1080 | .0336 | 9.2553 | .9664 | 50 | 1.4632 |
| 0.1105 | 20 | .1103 | .0426 | .9939 | .9973 | .1110 | .0453 | 9.0098 | .9547 | 40 | 1.4603 |
| 0.1134 | 30 | .1132 | .0539 | .9936 | .9972 | .1139 | .0567 | 8.7769 | .9433 | 30 | 1.4574 |
| 0.1164 | 40 | .1161 | .0648 | .9932 | .9971 | .1169 | .0678 | 8.5555 | .9322 | 20 | 1.4544 |
| 0.1193 | 50 | .1190 | .0755 | .9929 | .9969 | .1198 | .0786 | 8.3450 | .9214 | 10 | 1.4515 |
| 0.1222 | 7° 00′ | .1219 | 9.0859 | .9925 | 9.9968 | .1228 | 9.0891 | 8.1443 | 0.9109 | 83° 00′ | 1.4486 |
| 0.1251 | 10 | .1248 | .0961 | .9922 | .9966 | .1257 | .0995 | 7.9530 | .9005 | 50 | 1.4457 |
| 0.1280 | 20 | .1276 | .1060 | .9918 | .9964 | .1287 | .1096 | 7.7704 | .8904 | 40 | 1.4428 |
| 0.1309 | 30 | .1305 | .1157 | .9914 | .9963 | .1317 | .1194 | 7.5958 | .8806 | 30 | 1.4399 |
| 0.1338 | 40 | .1334 | .1252 | .9911 | .9961 | .1346 | .1291 | 7.4287 | .8709 | 20 | 1.4370 |
| 0.1367 | 50 | .1363 | .1345 | .9907 | .9959 | .1376 | .1385 | 7.2687 | .8615 | 10 | 1.4341 |
| 0.1396 | 8° 00′ | .1392 | 9.1436 | .9903 | 9.9958 | .1405 | 9.1478 | 7.1154 | 0.8522 | 82° 00′ | 1.4312 |
| 0.1425 | 10 | .1421 | .1525 | .9899 | .9956 | .1435 | .1569 | 6.9682 | .8431 | 50 | 1.4283 |
| 0.1454 | 20 | .1449 | .1612 | .9894 | .9954 | .1465 | .1658 | 6.8269 | .8342 | 40 | 1.4254 |
| 0.1484 | 30 | .1478 | .1697 | .9890 | .9952 | .1495 | .1745 | 6 6912 | .8255 | 30 | 1.4224 |
| 0.1513 | 40 | .1507 | .1781 | .9886 | .9950 | .1524 | .1831 | 6.5606 | .8169 | 20 | 1.4195 |
| 0.1542 | 50 | .1536 | .1863 | .9881 | .9948 | .1554 | .1915 | 6.4348 | .8085 | 10 | 1.4166 |
| 0.1571 | 9° 00′ | .1564 | 9.1943 | .9877 | 9.9946 | .1584 | 9.1997 | 6.3138 | 0.8003 | 81° 00′ | 1.4137 |
| | | Nat. | Log. | Nat. | Log. | Nat. | Log. | Nat. | Log. | | |
| | | COSINES. | | SINES. | | COTANGENTS. | | TANGENTS. | | DEGREES. | RADIANS. |

## Trigonometric Functions.

| RADIANS. | DEGREES. | SINES. | | COSINES. | | TANGENTS. | | COTANGENTS. | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Nat. | Log. | Nat. | Log. | Nat. | Log. | Nat. | Log. | | |
| 0.1571 | 9° 00′ | .1564 | 9.1943 | .9877 | 9.9946 | .1584 | 9.1997 | 6.3138 | 0.8003 | 81° 00′ | 1.4137 |
| 0.1600 | 10 | .1593 | .2022 | .9872 | .9944 | .1614 | .2078 | 6.1970 | .7922 | 50 | 1.4108 |
| 0.1629 | 20 | .1622 | .2100 | .9868 | .9942 | .1644 | .2158 | 6.0844 | .7842 | 40 | 1.4079 |
| 0.1658 | 30 | .1650 | .2176 | .9863 | .9940 | .1673 | .2236 | 5.9758 | .7764 | 30 | 1.4050 |
| 0.1687 | 40 | .1679 | .2251 | .9858 | .9938 | .1703 | .2313 | 5.8708 | .7687 | 20 | 1.4021 |
| 0.1716 | 50 | .1708 | .2324 | .9853 | .9936 | .1733 | .2389 | 5.7694 | .7611 | 10 | 1.3992 |
| 0.1745 | 10° 00′ | .1736 | 9.2397 | .9848 | 9.9934 | .1763 | 9.2463 | 5.6713 | 0.7537 | 80° 00′ | 1.3963 |
| 0.1774 | 10 | .1765 | .2468 | .9843 | .9931 | .1793 | .2536 | 5.5764 | .7464 | 50 | 1.3934 |
| 0.1804 | 20 | .1794 | .2538 | .9838 | .9929 | .1823 | .2609 | 5.4845 | .7391 | 40 | 1.3904 |
| 0.1833 | 30 | .1822 | .2606 | .9833 | .9927 | .1853 | .2680 | 5.3955 | .7320 | 30 | 1.3875 |
| 0.1862 | 40 | .1851 | .2674 | .9827 | .9924 | .1883 | .2750 | 5.3093 | .7250 | 20 | 1.3846 |
| 0.1891 | 50 | .1880 | .2740 | .9822 | .9922 | .1914 | .2819 | 5.2257 | .7181 | 10 | 1.3817 |
| 0.1920 | 11° 00′ | .1908 | 9.2806 | .9816 | 9.9919 | .1944 | 9.2887 | 5.1446 | 0.7113 | 79° 00′ | 1.3788 |
| 0.1949 | 10 | .1937 | .2870 | .9811 | .9917 | .1974 | .2953 | 5.0658 | .7047 | 50 | 1.3759 |
| 0.1978 | 20 | .1965 | .2934 | .9805 | .9914 | .2004 | .3020 | 4.9894 | .6980 | 40 | 1.3730 |
| 0.2007 | 30 | .1994 | .2997 | .9799 | .9912 | .2035 | .3085 | 4.9152 | .6915 | 30 | 1.3701 |
| 0.2036 | 40 | .2022 | .3058 | .9793 | .9909 | .2065 | .3149 | 4.8430 | .6851 | 20 | 1.3672 |
| 0.2065 | 50 | .2051 | .3119 | .9787 | .9907 | .2095 | .3212 | 4.7729 | .6788 | 10 | 1.3643 |
| 0.2094 | 12° 00′ | .2079 | 9.3179 | .9781 | 9.9904 | .2126 | 9.3275 | 4.7046 | 0.6725 | 78° 00′ | 1.3614 |
| 0.2123 | 10 | .2108 | .3238 | .9775 | .9901 | .2156 | .3336 | 4.6382 | .6664 | 50 | 1.3584 |
| 0.2153 | 20 | .2136 | .3296 | .9769 | .9899 | .2186 | .3397 | 4.5736 | .6603 | 40 | 1.3555 |
| 0.2182 | 30 | .2164 | .3353 | .9763 | .9896 | .2217 | .3458 | 4.5107 | .6542 | 30 | 1.3526 |
| 0.2211 | 40 | .2193 | .3410 | .9757 | .9893 | .2247 | .3517 | 4.4494 | .6483 | 20 | 1.3497 |
| 0.2240 | 50 | .2221 | .3466 | .9750 | .9890 | .2278 | .3576 | 4.3897 | .6424 | 10 | 1.3468 |
| 0.2269 | 13° 00′ | .2250 | 9.3521 | .9744 | 9.9887 | .2309 | 9.3634 | 4.3315 | 0.6366 | 77° 00′ | 1.3439 |
| 0.2298 | 10 | .2278 | .3575 | .9737 | .9884 | .2339 | .3691 | 4.2747 | .6309 | 50 | 1.3410 |
| 0.2327 | 20 | .2306 | .3629 | .9730 | .9881 | .2370 | .3748 | 4.2193 | .6252 | 40 | 1.3381 |
| 0.2356 | 30 | .2334 | .3682 | .9724 | .9878 | .2401 | .3804 | 4.1653 | .6196 | 30 | 1.3352 |
| 0.2385 | 40 | .2363 | .3734 | .9717 | .9875 | .2432 | .3859 | 4.1126 | .6141 | 20 | 1.3323 |
| 0.2414 | 50 | .2391 | .3786 | .9710 | .9872 | .2462 | .3914 | 4.0611 | .6086 | 10 | 1.3294 |
| 0.2443 | 14° 00′ | .2419 | 9.3837 | .9703 | 9.9869 | .2493 | 9.3968 | 4.0108 | 0.6032 | 76° 00′ | 1.3265 |
| 0.2473 | 10 | .2447 | .3887 | .9696 | .9866 | .2524 | .4021 | 3.9617 | .5979 | 50 | 1.3235 |
| 0.2502 | 20 | .2476 | .3937 | .9689 | .9863 | .2555 | .4074 | 3.9136 | .5926 | 40 | 1.3206 |
| 0.2531 | 30 | .2504 | .3986 | .9681 | .9859 | .2586 | .4127 | 3.8667 | .5873 | 30 | 1.3177 |
| 0.2560 | 40 | .2532 | .4035 | .9674 | .9856 | .2617 | .4178 | 3.8208 | .5822 | 20 | 1.3148 |
| 0.2589 | 50 | .2560 | .4083 | .9667 | .9853 | .2648 | .4230 | 3.7760 | .5770 | 10 | 1.3119 |
| 0.2618 | 15° 00′ | .2588 | 9.4130 | .9659 | 9.9849 | .2679 | 9.4281 | 3.7321 | 0.5719 | 75° 00′ | 1.3090 |
| 0.2647 | 10 | .2616 | .4177 | .9652 | .9846 | .2711 | .4331 | 3.6891 | .5669 | 50 | 1.3061 |
| 0.2676 | 20 | .2644 | .4223 | .9644 | .9843 | .2742 | .4381 | 3.6470 | .5619 | 40 | 1.3032 |
| 0.2705 | 30 | .2672 | .4269 | .9636 | .9839 | .2773 | .4430 | 3.6059 | .5570 | 30 | 1.3003 |
| 0.2734 | 40 | .2700 | .4314 | .9628 | .9836 | .2805 | .4479 | 3.5656 | .5521 | 20 | 1.2974 |
| 0.2763 | 50 | .2728 | .4359 | .9621 | .9832 | .2836 | .4527 | 3.5261 | .5473 | 10 | 1.2945 |
| 0.2793 | 16° 00′ | .2756 | 9.4403 | .9613 | 9.9828 | .2867 | 9.4575 | 3.4874 | 0.5425 | 74° 00′ | 1.2915 |
| 0.2822 | 10 | .2784 | .4447 | .9605 | .9825 | .2899 | .4622 | 3.4495 | .5378 | 50 | 1.2886 |
| 0.2851 | 20 | .2812 | .4491 | .9596 | .9821 | .2931 | .4669 | 3.4124 | .5331 | 40 | 1.2857 |
| 0.2880 | 30 | .2840 | .4533 | .9588 | .9817 | .2962 | .4716 | 3.3759 | .5284 | 30 | 1.2828 |
| 0.2909 | 40 | .2868 | .4576 | .9580 | .9814 | .2994 | .4762 | 3.3402 | .5238 | 20 | 1.2799 |
| 0.2938 | 50 | .2896 | .4618 | .9572 | .9810 | .3026 | .4808 | 3.3052 | .5192 | 10 | 1.2770 |
| 0.2967 | 17° 00′ | .2924 | 9.4659 | .9563 | 9.9806 | .3057 | 9.4853 | 3.2709 | 0.5147 | 73° 00 | 1.2741 |
| 0.2996 | 10 | .2952 | .4700 | .9555 | .9802 | .3089 | .4898 | 3.2371 | .5102 | 50 | 1.2712 |
| 0.3025 | 20 | .2979 | .4741 | .9546 | .9798 | .3121 | .4943 | 3.2041 | .5057 | 40 | 1.2683 |
| 0.3054 | 30 | .3007 | .4781 | .9537 | .9794 | .3153 | .4987 | 3.1716 | .5013 | 30 | 1.2654 |
| 0.3083 | 40 | .3035 | .4821 | .9528 | .9790 | .3185 | .5031 | 3.1397 | .4969 | 20 | 1.2625 |
| 0.3113 | 50 | .3062 | .4861 | .9520 | .9786 | .3217 | .5075 | 3.1084 | .4925 | 10 | 1.2595 |
| 0.3142 | 18° 00′ | .3090 | 9.4900 | .9511 | 9.9782 | .3249 | 9.5118 | 3.0777 | 0.4882 | 72° 00′ | 1.2566 |
| | | Nat. | Log. | Nat. | Log. | Nat. | Log. | Nat. | Log. | | |
| | | COSINES. | | SINES. | | COTANGENTS. | | TANGENTS. | | DEGREES. | RADIANS. |

## Trigonometric Functions.

| RADIANS. | DEGREES. | SINES. | | COSINES. | | TANGENTS. | | COTANGENTS. | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Nat. | Log. | Nat. | Log. | Nat. | Log. | Nat. | Log. | | |
| 0.3142 | 18° 00′ | .3090 | 9.4900 | .9511 | 9.9782 | .3249 | 9.5118 | 3 0777 | 0.4882 | 72° 00′ | 1.2566 |
| 0.3171 | 10 | .3118 | .4939 | .9502 | .9778 | .3281 | .5161 | 3.0475 | .4839 | 50 | 1.2537 |
| 0.3200 | 20 | .3145 | .4977 | .9492 | .9774 | .3314 | .5203 | 3.0178 | .4797 | 40 | 1.2508 |
| 0.3229 | 30 | .3173 | .5015 | .9483 | .9770 | .3346 | .5245 | 2 9887 | .4755 | 30 | 1.2479 |
| 0.3258 | 40 | .3201 | .5052 | .9474 | .9765 | .3378 | .5287 | 2 9600 | .4713 | 20 | 1 2450 |
| 0.3287 | 50 | .3228 | .5090 | .9465 | .9761 | .3411 | .5329 | 2.9319 | .4671 | 10 | 1.2421 |
| 0.3316 | 19° 00′ | .3256 | 9.5126 | .9455 | 9.9757 | .3443 | 9.5370 | 2.9042 | 0.4630 | 71° 00′ | 1.2392 |
| 0.3345 | 10 | .3283 | .5163 | .9446 | .9752 | .3476 | .5411 | 2.8770 | .4589 | 50 | 1.2363 |
| 0.3374 | 20 | .3311 | .5199 | .9436 | .9748 | .3508 | .5451 | 2.8502 | .4549 | 40 | 1.2334 |
| 0.3403 | 30 | .3338 | .5235 | .9426 | .9743 | .3541 | .5491 | 2.8239 | .4509 | 30 | 1.2305 |
| 0.3432 | 40 | .3365 | .5270 | .9417 | .9739 | .3574 | .5531 | 2.7980 | .4469 | 20 | 1.2275 |
| 0.3462 | 50 | .3393 | .5306 | .9407 | .9734 | .3607 | .5571 | 2.7725 | .4429 | 10 | 1.2246 |
| 0.3491 | 20° 00′ | .3420 | 9.5341 | .9397 | 9.9730 | .3640 | 9.5611 | 2.7475 | 0.4389 | 70° 00′ | 1.2217 |
| 0.3520 | 10 | .3448 | .5375 | .9387 | .9725 | .3673 | .5650 | 2.7228 | .4350 | 50 | 1.2188 |
| 0.3549 | 20 | .3475 | .5409 | .9377 | .9721 | .3706 | .5689 | 2.6985 | .4311 | 40 | 1.2159 |
| 0.3578 | 30 | .3502 | .5443 | .9367 | .9716 | .3739 | .5727 | 2.6746 | .4273 | 30 | 1.2130 |
| 0.3607 | 40 | .3529 | .5477 | .9356 | .9711 | .3772 | .5766 | 2.6511 | .4234 | 20 | 1.2101 |
| 0.3636 | 50 | .3557 | .5510 | .9346 | .9706 | .3805 | .5804 | 2.6279 | .4196 | 10 | 1.2072 |
| 0.3665 | 21° 00′ | .3584 | 9.5543 | .9336 | 9.9702 | .3839 | 9.5842 | 2.6051 | 0.4158 | 69° 00′ | 1.2043 |
| 0.3694 | 10 | .3611 | .5576 | .9325 | .9697 | .3872 | .5879 | 2.5826 | .4121 | 50 | 1.2014 |
| 0.3723 | 20 | .3638 | .5609 | .9315 | .9692 | .3906 | .5917 | 2.5605 | .4083 | 40 | 1.1985 |
| 0.3752 | 30 | .3565 | .5641 | .9304 | .9687 | .3939 | .5954 | 2 5386 | .4046 | 30 | 1.1956 |
| 0.3782 | 40 | .3692 | .5673 | .9293 | .9682 | .3973 | .5991 | 2.5172 | .4009 | 20 | 1.1926 |
| 0.3811 | 50 | .3719 | .5704 | .9283 | .9677 | .4006 | .6028 | 2.4960 | .3972 | 10 | 1.1897 |
| 0.3840 | 22° 00′ | .3746 | 9.5736 | .9272 | 9.9672 | .4040 | 9.6064 | 2.4751 | 0.3936 | 68° 00′ | 1.1868 |
| 0.3869 | 10 | .3773 | .5767 | .9261 | .9667 | .4074 | .6100 | 2.4545 | .3900 | 50 | 1.1839 |
| 0.3898 | 20 | .3800 | .5798 | .9250 | .9661 | .4108 | .6136 | 2 4342 | .3864 | 40 | 1.1810 |
| 0.3927 | 30 | .3827 | .5828 | .9239 | .9656 | .4142 | .6172 | 2.4142 | .3828 | 30 | 1.1781 |
| 0.3956 | 40 | .3854 | .5859 | .9228 | .9651 | .4176 | .6208 | 2.3945 | .3792 | 20 | 1.1752 |
| 0.3985 | 50 | .3881 | .5889 | .9216 | .9646 | .4210 | .6243 | 2 3750 | .3757 | 10 | 1.1723 |
| 0.4014 | 23° 00′ | .3907 | 9.5919 | .9205 | 9.9640 | .4245 | 9.6279 | 2.3559 | 0.3721 | 67° 00′ | 1.1694 |
| 0.4043 | 10 | .3934 | .5948 | .9194 | .9635 | .4279 | .6314 | 2 3369 | .3686 | 50 | 1.1665 |
| 0.4072 | 20 | .3961 | .5978 | .9182 | .9629 | .4314 | .6348 | 2.3183 | .3652 | 40 | 1.1636 |
| 0.4102 | 30 | .3987 | .6007 | .9171 | .9624 | .4348 | .6383 | 2.2998 | .3617 | 30 | 1.1606 |
| 0.4131 | 40 | .4014 | .6036 | .9159 | .9618 | .4383 | .6417 | 2 2817 | .3583 | 20 | 1.1577 |
| 0.4160 | 50 | .4041 | .6065 | .9147 | .9613 | .4417 | .6452 | 2.2637 | .3548 | 10 | 1.1548 |
| 0.4189 | 24° 00′ | .4067 | 9.6093 | .9135 | 9.9607 | .4452 | 9.6486 | 2 2460 | 0.3514 | 66° 00′ | 1.1519 |
| 0.4218 | 10 | .4094 | .6121 | .9124 | .9602 | .4487 | .6520 | 2.2286 | .3480 | 50 | 1.1490 |
| 0.4247 | 20 | .4120 | .6149 | .9112 | .9596 | .4522 | .6553 | 2.2113 | .3447 | 40 | 1.1461 |
| 0.4276 | 30 | .4147 | .6177 | .9100 | .9590 | .4557 | .6587 | 2.1943 | .3413 | 30 | 1.1432 |
| 0.4305 | 40 | .4173 | .6205 | .9088 | .9584 | .4592 | .6620 | 2.1775 | .3380 | 20 | 1.1403 |
| 0.4334 | 50 | .4200 | .6232 | .9075 | .9579 | .4628 | .6654 | 2.1609 | .3346 | 10 | 1.1374 |
| 0.4363 | 25° 00′ | .4226 | 9.6259 | .9063 | 9.9573 | .4663 | 9.6687 | 2 1445 | 0.3313 | 65° 00′ | 1.1345 |
| 0.4392 | 10 | .4253 | .6286 | .9051 | .9567 | .4699 | .6720 | 2.1283 | .3280 | 50 | 1.1316 |
| 0.4422 | 20 | .4279 | .6313 | .9038 | .9561 | .4734 | .6752 | 2.1123 | .3248 | 40 | 1.1286 |
| 0.4451 | 30 | .4305 | .6340 | .9026 | .9555 | .4770 | .6785 | 2.0965 | .3215 | 30 | 1.1257 |
| 0.4480 | 40 | .4331 | .6366 | .9013 | .9549 | .4806 | .6817 | 2.0809 | .3183 | 20 | 1.1228 |
| 0.4509 | 50 | .4358 | .6392 | .9001 | .9543 | .4841 | .6850 | 2.0655 | .3150 | 10 | 1.1199 |
| 0.4538 | 26° 00′ | .4384 | 9.6418 | .8988 | 9.9537 | .4877 | 9.6882 | 2.0503 | 0.3118 | 64° 00′ | 1.1170 |
| 0.4567 | 10 | .4410 | .6444 | .8975 | .9530 | .4913 | .6914 | 2 0353 | .3086 | 50 | 1.1141 |
| 0.4596 | 20 | .4436 | .6470 | .8962 | .9524 | .4950 | .6946 | 2.0204 | .3054 | 40 | 1.1112 |
| 0.4625 | 30 | .4462 | .6495 | .8949 | .9518 | .4986 | .6977 | 2.0057 | .3023 | 30 | 1.1083 |
| 0.4654 | 40 | .4488 | .6521 | .8936 | .9512 | .5022 | .7009 | 1.9912 | .2991 | 20 | 1.1054 |
| 0.4683 | 50 | .4514 | .6546 | .8923 | .9505 | .5059 | .7040 | 1.9768 | .2960 | 10 | 1.1025 |
| 0.4712 | 27° 00′ | .4540 | 9.6570 | .8910 | 9.9499 | .5095 | 9.7072 | 1.9626 | 0.2928 | 63° 00′ | 1.0996 |
| | | Nat. | Log. | Nat. | Log. | Nat. | Log. | Nat. | Log. | | |
| | | COSINES. | | SINES. | | COTANGENTS. | | TANGENTS. | | DEGREES. | RADIANS. |

## Trigonometric Functions.

| RADIANS. | DEGREES. | SINES. | | COSINES. | | TANGENTS. | | COTANGENTS. | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Nat. | Log. | Nat. | Log. | Nat. | Log. | Nat. | Log. | | |
| 0.4712 | 27° 00′ | .4540 | 9.6570 | .8910 | 9.9499 | .5095 | 9.7072 | 1.9626 | 0.2928 | 63° 00′ | 1.0996 |
| 0.4741 | 10 | .4566 | .6595 | .8897 | .9492 | .5132 | .7103 | 1.9486 | .2897 | 50 | 1.0966 |
| 0.4771 | 20 | .4592 | .6620 | .8884 | .9486 | .5169 | .7134 | 1.9347 | .2866 | 40 | 1.0937 |
| 0.4800 | 30 | .4617 | .6644 | .8870 | .9479 | .5206 | .7165 | 1.9210 | .2835 | 30 | 1.0908 |
| 0.4829 | 40 | .4643 | .6668 | .8857 | .9473 | .5243 | .7196 | 1.9074 | .2804 | 20 | 1.0879 |
| 0.4858 | 50 | .4669 | .6692 | .8843 | .9466 | .5280 | .7226 | 1.8940 | .2774 | 10 | 1.0850 |
| 0.4887 | 28° 00′ | .4695 | 9.6716 | .8829 | 9.9459 | .5317 | 9.7257 | 1.8807 | 0.2743 | 62° 00′ | 1.0821 |
| 0.4916 | 10 | .4720 | .6740 | .8816 | .9453 | .5354 | .7287 | 1.8676 | .2713 | 50 | 1.0792 |
| 0.4945 | 20 | .4746 | .6763 | .8802 | .9446 | .5392 | .7317 | 1.8546 | .2683 | 40 | 1.0763 |
| 0.4974 | 30 | .4772 | .6787 | .8788 | .9439 | .5430 | .7348 | 1.8418 | .2652 | 30 | 1.0734 |
| 0.5003 | 40 | .4797 | .6810 | .8774 | .9432 | .5467 | .7378 | 1.8291 | .2622 | 20 | 1.0705 |
| 0.5032 | 50 | .4823 | .6833 | .8760 | .9425 | .5505 | .7408 | 1.8165 | .2592 | 10 | 1.0676 |
| 0.5061 | 29° 00′ | .4848 | 9.6856 | .8746 | 9.9418 | .5543 | 9.7438 | 1.8040 | 0.2562 | 61° 00′ | 1.0647 |
| 0.5091 | 10 | .4874 | .6878 | .8732 | .9411 | .5581 | .7467 | 1.7917 | .2533 | 50 | 1.0617 |
| 0.5120 | 20 | .4899 | .6901 | .8718 | .9404 | .5619 | .7497 | 1.7796 | .2503 | 40 | 1.0588 |
| 0.5149 | 30 | .4924 | .6923 | .8704 | .9397 | .5658 | .7526 | 1.7675 | .2474 | 30 | 1.0559 |
| 0.5178 | 40 | .4950 | .6946 | .8689 | .9390 | .5696 | .7556 | 1.7556 | .2444 | 20 | 1.0530 |
| 0.5207 | 50 | .4975 | .6968 | .8675 | .9383 | .5735 | .7585 | 1.7437 | .2415 | 10 | 1.0501 |
| 0.5236 | 30° 00′ | .5000 | 9.6990 | .8660 | 9.9375 | .5774 | 9.7614 | 1.7321 | 0.2386 | 60° 00′ | 1.0472 |
| 0.5265 | 10 | .5025 | .7012 | .8646 | .9368 | .5812 | .7644 | 1.7205 | .2356 | 50 | 1.0443 |
| 0.5294 | 20 | .5050 | .7033 | .8631 | .9361 | .5851 | .7673 | 1.7090 | .2327 | 40 | 1.0414 |
| 0.5323 | 30 | .5075 | .7055 | .8616 | .9353 | .5890 | .7701 | 1.6977 | .2299 | 30 | 1.0385 |
| 0.5352 | 40 | .5100 | .7076 | .8601 | .9346 | .5930 | .7730 | 1.6864 | .2270 | 20 | 1.0356 |
| 0.5381 | 50 | .5125 | .7097 | .8587 | .9338 | .5969 | .7759 | 1.6753 | .2241 | 10 | 1.0327 |
| 0.5411 | 31° 00′ | .5150 | 9.7118 | .8572 | 9.9331 | .6009 | 9.7788 | 1.6643 | 0.2212 | 59° 00′ | 1.0297 |
| 0.5440 | 10 | .5175 | .7139 | .8557 | .9323 | .6048 | .7816 | 1.6534 | .2184 | 50 | 1.0268 |
| 0.5469 | 20 | .5200 | .7160 | .8542 | .9315 | .6088 | .7845 | 1.6426 | .2155 | 40 | 1.0239 |
| 0.5498 | 30 | .5225 | .7181 | .8526 | .9308 | .6128 | .7873 | 1.6319 | .2127 | 30 | 1.0210 |
| 0.5527 | 40 | .5250 | .7201 | .8511 | .9300 | .6168 | .7902 | 1.6212 | .2098 | 20 | 1.0181 |
| 0.5556 | 50 | .5275 | .7222 | .8496 | .9292 | .6208 | .7930 | 1.6107 | .2070 | 10 | 1.0152 |
| 0.5585 | 32° 00′ | .5299 | 9.7242 | .8480 | 9.9284 | .6249 | 9.7958 | 1.6003 | 0.2042 | 58° 00′ | 1.0123 |
| 0.5614 | 10 | .5324 | .7262 | .8465 | .9276 | .6289 | .7986 | 1.5900 | .2014 | 50 | 1.0094 |
| 0.5643 | 20 | .5348 | .7282 | .8450 | .9268 | .6330 | .8014 | 1.5798 | .1986 | 40 | 1.0065 |
| 0.5672 | 30 | .5373 | .7302 | .8434 | .9260 | .6371 | .8042 | 1.5697 | .1958 | 30 | 1.0036 |
| 0.5701 | 40 | .5398 | .7322 | .8418 | .9252 | .6412 | .8070 | 1.5597 | .1930 | 20 | 1.0007 |
| 0.5730 | 50 | .5422 | .7342 | .8403 | .9244 | .6453 | .8097 | 1.5497 | .1903 | 10 | 0.9977 |
| 0.5760 | 33° 00′ | .5446 | 9.7361 | .8387 | 9.9236 | .6494 | 9.8125 | 1.5399 | 0.1875 | 57° 00′ | 0.9948 |
| 0.5789 | 10 | .5471 | .7380 | .8371 | .9228 | .6536 | .8153 | 1.5301 | .1847 | 50 | 0.9919 |
| 0.5818 | 20 | .5495 | .7400 | .8355 | .9219 | .6577 | .8180 | 1.5204 | .1820 | 40 | 0.9890 |
| 0.5847 | 30 | .5519 | .7419 | .8339 | .9211 | .6619 | .8208 | 1.5108 | .1792 | 30 | 0.9861 |
| 0.5876 | 40 | .5544 | .7438 | .8323 | .9203 | .6661 | .8235 | 1.5013 | .1765 | 20 | 0.9832 |
| 0.5905 | 50 | .5568 | .7457 | .8307 | .9194 | .6703 | .8263 | 1.4919 | .1737 | 10 | 0.9803 |
| 0.5934 | 34° 00′ | .5592 | 9.7476 | .8290 | 9.9186 | .6745 | 9.8290 | 1.4826 | 0.1710 | 56° 00′ | 0.9774 |
| 0.5963 | 10 | .5616 | .7494 | .8274 | .9177 | .6787 | .8317 | 1.4733 | .1683 | 50 | 0.9745 |
| 0.5992 | 20 | .5640 | .7513 | .8258 | .9169 | .6830 | .8344 | 1.4641 | .1656 | 40 | 0.9716 |
| 0.6021 | 30 | .5664 | .7531 | .8241 | .9160 | 6873 | .8371 | 1.4550 | .1629 | 30 | 0.9687 |
| 0.6050 | 40 | .5688 | .7550 | .8225 | .9151 | .6916 | .8398 | 1.4460 | .1602 | 20 | 0.9657 |
| 0.6080 | 50 | .5712 | .7568 | .8208 | .9142 | .6959 | .8425 | 1.4370 | .1575 | 10 | 0.9628 |
| 0.6109 | 35° 00′ | .5736 | 9.7586 | .8192 | 9.9134 | .7002 | 9.8452 | 1.4281 | 0.1548 | 55° 00′ | 0.9599 |
| 0.6138 | 10 | .5760 | .7604 | .8175 | .9125 | .7046 | .8479 | 1.4193 | .1521 | 50 | 0.9570 |
| 0.6167 | 20 | .5783 | .7622 | .8158 | .9116 | .7089 | .8506 | 1.4106 | .1494 | 40 | 0.9541 |
| 0.6196 | 30 | .5807 | .7640 | .8141 | .9107 | .7133 | .8533 | 1.4019 | .1467 | 30 | 0.9512 |
| 0.6225 | 40 | .5831 | .7657 | .8124 | .9098 | .7177 | .8559 | 1.3934 | .1441 | 20 | 0.9483 |
| 0.6254 | 50 | .5854 | 7675 | .8107 | .9089 | .7221 | .8586 | 1.3848 | .1414 | 10 | 0.9454 |
| 0.6283 | 36° 00′ | .5878 | 9.7692 | .8090 | 9.9080 | .7265 | 9.8613 | 1.3764 | 0.1387 | 54° 00′ | 0.9425 |
| | | Nat. | Log. | Nat. | Log. | Nat. | Log. | Nat. | Log. | | |
| | | COSINES. | | SINES. | | COTANGENTS. | | TANGENTS. | | DEGREES. | RADIANS. |

## Trigonometric Functions.

| RADIANS. | DEGREES. | SINES. | | COSINES. | | TANGENTS. | | COTANGENTS. | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Nat. | Log. | Nat. | Log. | Nat. | Log. | Nat. | Log. | | |
| 0.6283 | 36° 00′ | .5878 | 9.7692 | .8090 | 9.9080 | .7265 | 9.8613 | 1.3764 | 0.1387 | 54° 00′ | 0.9425 |
| 0.6312 | 10 | .5901 | .7710 | .8073 | .9070 | .7310 | .8639 | 1.3680 | .1361 | 50 | 0.9396 |
| 0.6341 | 20 | .5925 | .7727 | .8056 | .9061 | .7355 | .8666 | 1.3597 | .1334 | 40 | 0.9367 |
| 0.6370 | 30 | .5948 | .7744 | .8039 | .9052 | .7400 | .8692 | 1.3514 | .1308 | 30 | 0.9338 |
| 0.6400 | 40 | .5972 | .7761 | .8021 | .9042 | .7445 | .8718 | 1.3432 | .1282 | 20 | 0.9308 |
| 0.6429 | 50 | .5995 | .7778 | .8004 | .9033 | .7490 | .8745 | 1.3351 | .1255 | 10 | 0.9279 |
| 0.6458 | 37° 00′ | .6018 | 9.7795 | .7986 | 9.9023 | .7536 | 9.8771 | 1.3270 | 0.1229 | 53° 00′ | 0.9250 |
| 0.6487 | 10 | .6041 | .7811 | .7969 | .9014 | .7581 | .8797 | 1.3190 | .1203 | 50 | 0.9221 |
| 0.6516 | 20 | .6065 | .7828 | .7951 | .9004 | .7627 | .8824 | 1.3111 | .1176 | 40 | 0.9192 |
| 0.6545 | 30 | .6088 | .7844 | .7934 | .8995 | .7673 | .8850 | 1.3032 | .1150 | 30 | 0.9163 |
| 0.6574 | 40 | .6111 | .7861 | .7916 | .8985 | .7720 | .8876 | 1.2954 | .1124 | 20 | 0.9134 |
| 0.6603 | 50 | .6134 | .7877 | .7898 | .8975 | .7766 | .8902 | 1.2876 | .1098 | 10 | 0.9105 |
| 0.6632 | 38° 00′ | .6157 | 9.7893 | .7880 | 9.8965 | .7813 | 9.8928 | 1.2799 | 0.1072 | 52° 00′ | 0.9076 |
| 0.6661 | 10 | .6180 | .7910 | .7862 | .8955 | .7860 | .8954 | 1.2723 | .1046 | 50 | 0.9047 |
| 0.6690 | 20 | .6202 | .7926 | .7844 | .8945 | .7907 | .8980 | 1.2647 | .1020 | 40 | 0.9018 |
| 0.6720 | 30 | .6225 | .7941 | .7826 | .8935 | .7954 | .9006 | 1.2572 | .0994 | 30 | 0.8988 |
| 0.6749 | 40 | .6248 | .7957 | .7808 | .8925 | .8002 | .9032 | 1.2497 | .0968 | 20 | 0.8959 |
| 0.6778 | 50 | .6271 | .7973 | .7790 | .8915 | .8050 | .9058 | 1.2423 | .0942 | 10 | 0.8930 |
| 0.6807 | 39° 00′ | .6293 | 9.7989 | .7771 | 9.8905 | .8098 | 9.9084 | 1.2349 | 0.0916 | 51° 00′ | 0.8901 |
| 0.6836 | 10 | .6316 | .8004 | .7753 | .8895 | .8146 | .9110 | 1.2276 | .0890 | 50 | 0.8872 |
| 0.6865 | 20 | .6338 | .8020 | .7735 | .8884 | .8195 | .9135 | 1.2203 | .0865 | 40 | 0.8843 |
| 0.6894 | 30 | .6361 | .8035 | .7716 | .8874 | .8243 | 9161 | 1.2131 | .0839 | 30 | 0.8814 |
| 0.6923 | 40 | .6383 | .8050 | .7698 | .8864 | .8292 | 9187 | 1.2059 | .0813 | 20 | 0.8785 |
| 0.6952 | 50 | .6406 | .8066 | .7679 | .8853 | .8342 | .9212 | 1.1988 | .0788 | 10 | 0.8756 |
| 0.6981 | 40° 00′ | .6428 | 9.8081 | .7660 | 9.8843 | .8391 | 9.9238 | 1.1918 | 0.0762 | 50° 00′ | 0.8727 |
| 0.7010 | 10 | .6450 | .8096 | .7642 | .8832 | .8441 | .9264 | 1.1847 | .0736 | 50 | 0.8698 |
| 0.7039 | 20 | .6472 | .8111 | .7623 | .8821 | .8491 | .9289 | 1.1778 | .0711 | 40 | 0.8668 |
| 0.7069 | 30 | .6494 | .8125 | .7604 | .8810 | .8541 | .9315 | 1.1708 | .0685 | 30 | 0.8639 |
| 0.7098 | 40 | .6517 | .8140 | .7585 | .8800 | .8591 | .9341 | 1.1640 | .0659 | 20 | 0.8610 |
| 0.7127 | 50 | .6539 | .8155 | .7566 | .8789 | .8642 | .9366 | 1.1571 | .0634 | 10 | 0.8581 |
| 0.7156 | 41° 00′ | .6561 | 9.8169 | .7547 | 9.8778 | .8693 | 9.9392 | 1.1504 | 0.0608 | 49° 00′ | 0.8552 |
| 0.7185 | 10 | .6583 | .8184 | .7528 | .8767 | .8744 | .9417 | 1.1436 | .0583 | 50 | 0.8523 |
| 0.7214 | 20 | .6604 | .8198 | .7509 | .8756 | .8796 | .9443 | 1.1369 | .0557 | 40 | 0.8494 |
| 0.7243 | 30 | .6626 | .8213 | .7490 | .8745 | .8847 | .9468 | 1.1303 | .0532 | 30 | 0.8465 |
| 0.7272 | 40 | .6648 | .8227 | .7470 | .8733 | .8899 | .9494 | 1.1237 | .0506 | 20 | 0.8436 |
| 0.7301 | 50 | .6670 | .8241 | .7451 | .8722 | .8952 | .9519 | 1.1171 | .0481 | 10 | 0.8407 |
| 0.7330 | 42° 00′ | .6691 | 9.8255 | .7431 | 9.8711 | .9004 | 9.9544 | 1.1106 | 0.0456 | 48° 00′ | 0.8378 |
| 0.7359 | 10 | .6713 | .8269 | .7412 | .8699 | .9057 | .9570 | 1.1041 | .0430 | 50 | 0.8348 |
| 0.7389 | 20 | .6734 | .8283 | .7392 | .8688 | .9110 | .9595 | 1.0977 | .0405 | 40 | 0.8319 |
| 0.7418 | 30 | .6756 | .8297 | .7373 | .8676 | .9163 | .9621 | 1.0913 | .0379 | 30 | 0.8290 |
| 0.7447 | 40 | .6777 | .8311 | .7353 | .8665 | .9217 | .9646 | 1.0850 | .0354 | 20 | 0.8261 |
| 0.7476 | 50 | .6799 | .8324 | .7333 | .8653 | .9271 | .9671 | 1.0786 | .0329 | 10 | 0.8232 |
| 0.7505 | 43° 00′ | .6820 | 9.8338 | .7314 | 9.8641 | .9325 | 9.9697 | 1.0724 | 0.0303 | 47° 00′ | 0.8203 |
| 0.7534 | 10 | .6841 | .8351 | .7294 | .8629 | .9380 | .9722 | 1.0661 | .0278 | 50 | 0.8174 |
| 0.7563 | 20 | .6862 | .8365 | .7274 | .8618 | .9435 | .9747 | 1.0599 | .0253 | 40 | 0.8145 |
| 0.7592 | 30 | .6884 | .8378 | .7254 | .8606 | .9490 | .9772 | 1.0538 | .0228 | 30 | 0.8116 |
| 0.7621 | 40 | .6905 | .8391 | .7234 | .8594 | .9545 | .9798 | 1.0477 | .0202 | 20 | 0.8087 |
| 0.7650 | 50 | .6926 | .8405 | .7214 | .8582 | .9601 | .9823 | 1.0416 | .0177 | 10 | 0.8058 |
| 0.7679 | 44° 00′ | .6947 | 9.8418 | .7193 | 9.8569 | .9657 | 9.9848 | 1.0355 | 0.0152 | 46° 00′ | 0.8029 |
| 0.7709 | 10 | .6967 | .8431 | .7173 | .8557 | .9713 | .9874 | 1.0295 | .0126 | 50 | 0.7999 |
| 0.7738 | 20 | .6988 | .8444 | .7153 | .8545 | .9770 | .9899 | 1.0235 | .0101 | 40 | 0.7970 |
| 0.7767 | 30 | .7009 | 8457 | .7133 | .8532 | .9827 | .9924 | 1.0176 | .0076 | 30 | 0.7941 |
| 0.7796 | 40 | .7030 | .8469 | .7112 | .8520 | .9884 | .9949 | 1.0117 | .0051 | 20 | 0.7912 |
| 0.7825 | 50 | .7050 | .8482 | .7092 | .8507 | .9942 | .9975 | 1.0058 | .0025 | 10 | 0.7883 |
| 0.7854 | 45° 00′ | .7071 | 9.8495 | .7071 | 9.8495 | 1.0000 | 0.0000 | 1.0000 | 0.0000 | 45° 00′ | 0.7854 |
| | | Nat. | Log. | Nat. | Log. | Nat. | Log. | Nat. | Log. | | |
| | | COSINES. | | SINES. | | COTANGENTS. | | TANGENTS. | | DEGREES. | RADIANS. |

## Equivalents of Radians in Degrees, Minutes, and Seconds of Arc.

| RADIANS. | EQUIVALENTS. | RADIANS. | EQUIVALENTS. |
|---|---|---|---|
| 0.0001 | 0° 0′ 20″.6 or 0°.005730 | 0.0600 | 3° 26′ 15″.9 or 3°.437747 |
| 0.0002 | 0° 0′ 41″.3 or 0°.011459 | 0.0700 | 4° 0′ 38″.5 or 4°.010705 |
| 0.0003 | 0° 1′ 01″.9 or 0°.017189 | 0.0800 | 4° 35′ 01″.2 or 4°.583662 |
| 0.0004 | 0° 1′ 22″.5 or 0°.022918 | 0.0900 | 5° 9′ 23″.8 or 5°.156620 |
| 0.0005 | 0° 1′ 43″.1 or 0°.028648 | 0.1000 | 5° 43′ 46″.5 or 5°.729578 |
| 0.0006 | 0° 2′ 03″.8 or 0°.034377 | 0.2000 | 11° 27′ 33″.0 or 11°.459156 |
| 0.0007 | 0° 2′ 24″.4 or 0°.040107 | 0.3000 | 17° 11′ 19″.4 or 17°.188734 |
| 0.0008 | 0° 2′ 45″.0 or 0°.045837 | 0.4000 | 22° 55′ 05″.9 or 22°.918312 |
| 0.0009 | 0° 3′ 05″.6 or 0°.051566 | 0.5000 | 28° 38′ 52″.4 or 28°.647890 |
| 0.0010 | 0° 3′ 26″.3 or 0°.057296 | 0.6000 | 34° 22′ 38″.9 or 34°.377468 |
| 0.0020 | 0° 6′ 52″.5 or 0°.114592 | 0.7000 | 40° 6′ 25″.4 or 40°.107046 |
| 0.0030 | 0° 10′ 18″.8 or 0°.171887 | 0.8000 | 45° 50′ 11″.8 or 45°.836624 |
| 0.0040 | 0° 13′ 45″.1 or 0°.229183 | 0.9000 | 51° 33′ 58″.3 or 51°.566202 |
| 0.0050 | 0° 17′ 11″.3 or 0°.286479 | 1.0000 | 57° 17′ 44″.8 or 57°.295780 |
| 0.0060 | 0° 20′ 37″.6 or 0°.343775 | 2.0000 | 114° 35′ 29″.6 or 114°.591559 |
| 0.0070 | 0° 24′ 03″.9 or 0°.401070 | 3.0000 | 171° 53′ 14″.4 or 171°.887339 |
| 0.0080 | 0° 27′ 30″.1 or 0°.458366 | 4.0000 | 229° 10′ 59″.2 or 229°.183118 |
| 0.0090 | 0° 30′ 56″.4 or 0°.515662 | 5.0000 | 286° 28′ 44″.0 or 286°.478898 |
| 0.0100 | 0° 34′ 22″.6 or 0°.572958 | 6.0000 | 343° 46′ 28″.8 or 343°.774677 |
| 0.0200 | 1° 8′ 45″.3 or 1°.145916 | 7.0000 | 401° 4′ 13″.6 or 401°.070457 |
| 0.0300 | 1° 43′ 07″.9 or 1°.718873 | 8.0000 | 458° 21′ 58″.4 or 458°.366236 |
| 0.0400 | 2° 17′ 30″.6 or 2°.291831 | 9.0000 | 515° 39′ 43″.3 or 515°.662016 |
| 0.0500 | 2° 51′ 53″.2 or 2°.864789 | 10.0000 | 572° 57′ 28″.1 or 572°.957795 |

## The Values in Circular Measure of Angles which are given in Degrees and Minutes.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1′ | 0.0003 | 9′ | 0.0026 | 3° | 0.0524 | 20° | 0.3491 | 100° | 1.7453 |
| 2′ | 0.0006 | 10′ | 0.0029 | 4° | 0.0698 | 30° | 0.5236 | 110° | 1.9199 |
| 3′ | 0.0009 | 20′ | 0.0058 | 5° | 0.0873 | 40° | 0.6981 | 120° | 2.0944 |
| 4′ | 0.0012 | 30′ | 0.0087 | 6° | 0.1047 | 50° | 0.8727 | 130° | 2.2689 |
| 5′ | 0.0015 | 40′ | 0.0116 | 7° | 0.1222 | 60° | 1.0472 | 140° | 2.4435 |
| 6′ | 0.0017 | 50′ | 0.0145 | 8° | 0.1396 | 70° | 1.2217 | 150° | 2.6180 |
| 7′ | 0.0020 | 1° | 0.0175 | 9° | 0.1571 | 80° | 1.3963 | 160° | 2.7925 |
| 8′ | 0.0023 | 2° | 0.0349 | 10° | 0.1745 | 90° | 1.5708 | 170° | 2.9671 |

## Square Roots of Numbers.

| $N$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Avg. diff. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1.0 | 1.000 | 1.005 | 1.010 | 1.015 | 1.020 | 1.025 | 1.030 | 1.034 | 1.039 | 1.044 | 5 |
| 1 | 1.049 | 1.054 | 1.058 | 1.063 | 1.068 | 1.072 | 1.077 | 1.082 | 1.086 | 1.091 | |
| 2 | 1.095 | 1.100 | 1.105 | 1.109 | 1.114 | 1.118 | 1.122 | 1.127 | 1.131 | 1.136 | 4 |
| 3 | 1.140 | 1.145 | 1.149 | 1.153 | 1.158 | 1.162 | 1.166 | 1.170 | 1.175 | 1.179 | |
| 4 | 1.183 | 1.187 | 1.192 | 1.196 | 1.200 | 1.204 | 1.208 | 1.212 | 1.217 | 1.221 | |
| 1.5 | 1.225 | 1.229 | 1.233 | 1.237 | 1.241 | 1.245 | 1.249 | 1.253 | 1.257 | 1.261 | |
| 6 | 1.265 | 1.269 | 1.273 | 1.277 | 1.281 | 1.285 | 1.288 | 1.292 | 1.296 | 1.300 | |
| 7 | 1.304 | 1.308 | 1.311 | 1.315 | 1.319 | 1.323 | 1.327 | 1.330 | 1.334 | 1.338 | |
| 8 | 1.342 | 1.345 | 1.349 | 1.353 | 1.356 | 1.360 | 1.364 | 1.367 | 1.371 | 1.375 | |
| 9 | 1.378 | 1.382 | 1.386 | 1.389 | 1.393 | 1.396 | 1.400 | 1.404 | 1.407 | 1.411 | |
| 2.0 | 1.414 | 1.418 | 1.421 | 1.425 | 1.428 | 1.432 | 1.435 | 1.439 | 1.442 | 1.446 | |
| 1 | 1.449 | 1.453 | 1.456 | 1.459 | 1.463 | 1.466 | 1.470 | 1.473 | 1.476 | 1.480 | 3 |
| 2 | 1.483 | 1.487 | 1.490 | 1.493 | 1.497 | 1.500 | 1.503 | 1.507 | 1.510 | 1.513 | |
| 3 | 1.517 | 1.520 | 1.523 | 1.526 | 1.530 | 1.533 | 1.536 | 1.539 | 1.543 | 1.546 | |
| 4 | 1.549 | 1.552 | 1.556 | 1.559 | 1.562 | 1.565 | 1.568 | 1.572 | 1.575 | 1.578 | |
| 2.5 | 1.581 | 1.584 | 1.587 | 1.591 | 1.594 | 1.597 | 1.600 | 1.603 | 1.606 | 1.609 | |
| 6 | 1.612 | 1.616 | 1.619 | 1.622 | 1.625 | 1.628 | 1.631 | 1.634 | 1.637 | 1.640 | |
| 7 | 1.643 | 1.646 | 1.649 | 1.652 | 1.655 | 1.658 | 1.661 | 1.664 | 1.667 | 1.670 | |
| 8 | 1.673 | 1.676 | 1.679 | 1.682 | 1.685 | 1.688 | 1.691 | 1.694 | 1.697 | 1.700 | |
| 9 | 1.703 | 1.706 | 1.709 | 1.712 | 1.715 | 1.718 | 1.720 | 1.723 | 1.726 | 1.729 | |
| 3.0 | 1.732 | 1.735 | 1.738 | 1.741 | 1.744 | 1.746 | 1.749 | 1.752 | 1.755 | 1.758 | |
| 1 | 1.761 | 1.764 | 1.766 | 1.769 | 1.772 | 1.775 | 1.778 | 1.780 | 1.783 | 1.786 | |
| 2 | 1.789 | 1.792 | 1.794 | 1.797 | 1.800 | 1.803 | 1.806 | 1.808 | 1.811 | 1.814 | |
| 3 | 1.817 | 1.819 | 1.822 | 1.825 | 1.828 | 1.830 | 1.833 | 1.836 | 1.838 | 1.841 | |
| 4 | 1.844 | 1.847 | 1.849 | 1.852 | 1.855 | 1.857 | 1.860 | 1.863 | 1.865 | 1.868 | |
| 3.5 | 1.871 | 1.873 | 1.876 | 1.879 | 1.881 | 1.884 | 1.887 | 1.889 | 1.892 | 1.895 | |
| 6 | 1.897 | 1.900 | 1.903 | 1.905 | 1.908 | 1.910 | 1.913 | 1.916 | 1.918 | 1.921 | |
| 7 | 1.924 | 1.926 | 1.929 | 1.931 | 1.934 | 1.936 | 1.939 | 1.942 | 1.944 | 1.947 | |
| 8 | 1.949 | 1.952 | 1.954 | 1.957 | 1.960 | 1.962 | 1.965 | 1.967 | 1.970 | 1.972 | |
| 9 | 1.975 | 1.977 | 1.980 | 1.982 | 1.985 | 1.987 | 1.990 | 1.992 | 1.995 | 1.997 | |
| 4.0 | 2.000 | 2.002 | 2.005 | 2.007 | 2.010 | 2.012 | 2.015 | 2.017 | 2.020 | 2.022 | |
| 1 | 2.025 | 2.027 | 2.030 | 2.032 | 2.035 | 2.037 | 2.040 | 2.042 | 2.045 | 2.047 | 2 |
| 2 | 2.049 | 2.052 | 2.054 | 2.057 | 2.059 | 2.062 | 2.064 | 2.066 | 2.069 | 2.071 | |
| 3 | 2.074 | 2.076 | 2.078 | 2.081 | 2.083 | 2.086 | 2.088 | 2.090 | 2.093 | 2.095 | |
| 4 | 2.098 | 2.100 | 2.102 | 2.105 | 2.107 | 2.110 | 2.112 | 2.114 | 2.117 | 2.119 | |
| 4.5 | 2.121 | 2.124 | 2.126 | 2.128 | 2.131 | 2.133 | 2.135 | 2.138 | 2.140 | 2.142 | |
| 6 | 2.145 | 2.147 | 2.149 | 2.152 | 2.154 | 2.156 | 2.159 | 2.161 | 2.163 | 2.166 | |
| 7 | 2.168 | 2.170 | 2.173 | 2.175 | 2.177 | 2.179 | 2.182 | 2.184 | 2.186 | 2.189 | |
| 8 | 2.191 | 2.193 | 2.195 | 2.198 | 2.200 | 2.202 | 2.205 | 2.207 | 2.209 | 2.211 | |
| 9 | 2.214 | 2.216 | 2.218 | 2.220 | 2.223 | 2.225 | 2.227 | 2.229 | 2.232 | 2.234 | |

$\sqrt{\pi} = 1.77245+$ $\quad 1/\sqrt{\pi} = 0.56419$ $\quad \sqrt{\pi/2} = 1.25331$ $\quad \sqrt{e} = 1.64872$

### Explanation of Table of Square Roots.

This table gives the values of $\sqrt{N}$ for values of $N$ from 1 to 100, correct to four figures. (Interpolated values may be in error by 1 in the fourth figure.)

**To find the square root of a number N outside the range from 1 to 100,** divide the digits of the number into blocks of two (beginning with the decimal point), and note that moving the decimal point **two** places in $N$ is equivalent to moving it **one place in the square root of** $N$. For example:

$$\sqrt{2.718} = 1.648; \quad \sqrt{271.8} = 16.48; \quad \sqrt{0.0002718} = 0.01648;$$

$$\sqrt{27.18} = 5.213; \quad \sqrt{2718} = 52.13; \quad \sqrt{0.002718} = 0.05213.$$

Square Roots.

| $N$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Avg. dif. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5.0 | 2.236 | 2.238 | 2.241 | 2.243 | 2.245 | 2.247 | 2.249 | 2.252 | 2.254 | 2.256 | 2 |
| 1 | 2.258 | 2.261 | 2.263 | 2.265 | 2.267 | 2.269 | 2.272 | 2.274 | 2.276 | 2.278 | |
| 2 | 2.280 | 2.283 | 2.285 | 2.287 | 2.289 | 2.291 | 2.293 | 2.296 | 2.298 | 2.300 | |
| 3 | 2.302 | 2.304 | 2.307 | 2.309 | 2.311 | 2.313 | 2.315 | 2.317 | 2.319 | 2.322 | |
| 4 | 2.324 | 2.326 | 2.328 | 2.330 | 2.332 | 2.335 | 2.337 | 2.339 | 2.341 | 2.343 | |
| 5.5 | 2.345 | 2.347 | 2.349 | 2.352 | 2.354 | 2.356 | 2.358 | 2.360 | 2.362 | 2.364 | |
| 6 | 2.366 | 2.369 | 2.371 | 2.373 | 2.375 | 2.377 | 2.379 | 2.381 | 2.383 | 2.385 | |
| 7 | 2.387 | 2.390 | 2.392 | 2.394 | 2.396 | 2.398 | 2.400 | 2.402 | 2.404 | 2.406 | |
| 8 | 2.408 | 2.410 | 2.412 | 2.415 | 2.417 | 2.419 | 2.421 | 2.423 | 2.425 | 2.427 | |
| 9 | 2.429 | 2.431 | 2.433 | 2.435 | 2.437 | 2.439 | 2.441 | 2.443 | 2.445 | 2.447 | |
| 6.0 | 2.449 | 2.452 | 2.454 | 2.456 | 2.458 | 2.460 | 2.462 | 2.464 | 2.466 | 2.468 | |
| 1 | 2.470 | 2.472 | 2.474 | 2.476 | 2.478 | 2.480 | 2.482 | 2.484 | 2.486 | 2.488 | |
| 2 | 2.490 | 2.492 | 2.494 | 2.496 | 2.498 | 2.500 | 2.502 | 2.504 | 2.506 | 2.508 | |
| 3 | 2.510 | 2.512 | 2.514 | 2.516 | 2.518 | 2.520 | 2.522 | 2.524 | 2.526 | 2.528 | |
| 4 | 2.530 | 2.532 | 2.534 | 2.536 | 2.538 | 2.540 | 2.542 | 2.544 | 2.546 | 2.548 | |
| 6.5 | 2.550 | 2.551 | 2.553 | 2.555 | 2.557 | 2.559 | 2.561 | 2.563 | 2.565 | 2.567 | |
| 6 | 2.569 | 2.571 | 2.573 | 2.575 | 2.577 | 2.579 | 2.581 | 2.583 | 2.585 | 2.587 | |
| 7 | 2.588 | 2.590 | 2.592 | 2.594 | 2.596 | 2.598 | 2.600 | 2.602 | 2.604 | 2.606 | |
| 8 | 2.608 | 2.610 | 2.612 | 2.613 | 2.615 | 2.617 | 2.619 | 2.621 | 2.623 | 2.625 | |
| 9 | 2.627 | 2.629 | 2.631 | 2.632 | 2.634 | 2.636 | 2.638 | 2.640 | 2.642 | 2.644 | |
| 7.0 | 2.646 | 2.648 | 2.650 | 2.651 | 2.653 | 2.655 | 2.657 | 2.659 | 2.661 | 2.663 | |
| 1 | 2.665 | 2.666 | 2.668 | 2.670 | 2.672 | 2.674 | 2.676 | 2.678 | 2.680 | 2.681 | |
| 2 | 2.683 | 2.685 | 2.687 | 2.689 | 2.691 | 2.693 | 2.694 | 2.696 | 2.698 | 2.700 | |
| 3 | 2.702 | 2.704 | 2.706 | 2.707 | 2.709 | 2.711 | 2.713 | 2.715 | 2.717 | 2.718 | |
| 4 | 2.720 | 2.722 | 2.724 | 2.726 | 2.728 | 2.729 | 2.731 | 2.733 | 2.735 | 2.737 | |
| 7.5 | 2.739 | 2.740 | 2.742 | 2.744 | 2.746 | 2.748 | 2.750 | 2.751 | 2.753 | 2.755 | |
| 6 | 2.757 | 2.759 | 2.760 | 2.762 | 2.764 | 2.766 | 2.768 | 2.769 | 2.771 | 2.773 | |
| 7 | 2.775 | 2.777 | 2.778 | 2.780 | 2.782 | 2.784 | 2.786 | 2.787 | 2.789 | 2.791 | |
| 8 | 2.793 | 2.795 | 2.796 | 2.798 | 2.800 | 2.802 | 2.804 | 2.805 | 2.807 | 2.809 | |
| 9 | 2.811 | 2.812 | 2.814 | 2.816 | 2.818 | 2.820 | 2.821 | 2.823 | 2.825 | 2.827 | |
| 8.0 | 2.828 | 2.830 | 2.832 | 2.834 | 2.835 | 2.837 | 2.839 | 2.841 | 2.843 | 2.844 | |
| 1 | 2.846 | 2.848 | 2.850 | 2.851 | 2.853 | 2.855 | 2.857 | 2.858 | 2.860 | 2.862 | |
| 2 | 2.864 | 2.865 | 2.867 | 2.869 | 2.871 | 2.872 | 2.874 | 2.876 | 2.877 | 2.879 | |
| 3 | 2.881 | 2.883 | 2.884 | 2.886 | 2.888 | 2.890 | 2.891 | 2.893 | 2.895 | 2.897 | |
| 4 | 2.898 | 2.900 | 2.902 | 2.903 | 2.905 | 2.907 | 2.909 | 2.910 | 2.912 | 2.914 | |
| 8.5 | 2.915 | 2.917 | 2.919 | 2.921 | 2.922 | 2.924 | 2.926 | 2.927 | 2.929 | 2.931 | |
| 6 | 2.933 | 2.934 | 2.936 | 2.938 | 2.939 | 2.941 | 2.943 | 2.944 | 2.946 | 2.948 | |
| 7 | 2.950 | 2.951 | 2.953 | 2.955 | 2.956 | 2.958 | 2.960 | 2.961 | 2.963 | 2.965 | |
| 8 | 2.966 | 2.968 | 2.970 | 2.972 | 2.973 | 2.975 | 2.977 | 2.978 | 2.980 | 2.982 | |
| 9 | 2.983 | 2.985 | 2.987 | 2.988 | 2.990 | 2.992 | 2.993 | 2.995 | 2.997 | 2.998 | |
| 9.0 | 3.000 | 3.002 | 3.003 | 3.005 | 3.007 | 3.008 | 3.010 | 3.012 | 3.013 | 3.015 | |
| 1 | 3.017 | 3.018 | 3.020 | 3.022 | 3.023 | 3.025 | 3.027 | 3.028 | 3.030 | 3.032 | |
| 2 | 3.033 | 3.035 | 3.036 | 3.038 | 3.040 | 3.041 | 3.043 | 3.045 | 3.046 | 3.048 | |
| 3 | 3.050 | 3.051 | 3.053 | 3.055 | 3.056 | 3.058 | 3.059 | 3.061 | 3.063 | 3.064 | |
| 4 | 3.066 | 3.068 | 3.069 | 3.071 | 3.072 | 3.074 | 3.076 | 3.077 | 3.079 | 3.081 | |
| 9.5 | 3.082 | 3.084 | 3.085 | 3.087 | 3.089 | 3.090 | 3.092 | 3.094 | 3.095 | 3.097 | |
| 6 | 3.098 | 3.100 | 3.102 | 3.103 | 3.105 | 3.106 | 3.108 | 3.110 | 3.111 | 3.113 | |
| 7 | 3.114 | 3.116 | 3.118 | 3.119 | 3.121 | 3.122 | 3.124 | 3.126 | 3.127 | 3.129 | |
| 8 | 3.130 | 3.132 | 3.134 | 3.135 | 3.137 | 3.138 | 3.140 | 3.142 | 3.143 | 3.145 | |
| 9 | 3.146 | 3.148 | 3.150 | 3.151 | 3.153 | 3.154 | 3.156 | 3.158 | 3.159 | 3.161 | |

Moving the decimal point TWO places in $N$ requires moving it ONE place in body of table.

## Square Roots.

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Avg. diff. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 10. | 3.162 | 3.178 | 3.194 | 3.209 | 3.225 | 3.240 | 3.256 | 3.271 | 3.286 | 3.302 | 16 |
| 1. | 3.317 | 3.332 | 3.347 | 3.362 | 3.376 | 3.391 | 3.406 | 3.421 | 3.435 | 3.450 | 15 |
| 2. | 3.464 | 3.479 | 3.493 | 3.507 | 3.521 | 3.536 | 3.550 | 3.564 | 3.578 | 3.592 | 14 |
| 3. | 3.606 | 3.619 | 3.633 | 3.647 | 3.661 | 3.674 | 3.688 | 3.701 | 3.715 | 3.728 | |
| 4. | 3.742 | 3.755 | 3.768 | 3.782 | 3.795 | 3.808 | 3.821 | 3.834 | 3.847 | 3.860 | 13 |
| 15. | 3.873 | 3.886 | 3.899 | 3.912 | 3.924 | 3.937 | 3.950 | 3.962 | 3.975 | 3.987 | |
| 6. | 4.000 | 4.012 | 4.025 | 4.037 | 4.050 | 4.062 | 4.074 | 4.087 | 4.099 | 4.111 | 12 |
| 7. | 4.123 | 4.135 | 4.147 | 4.159 | 4.171 | 4.183 | 4.195 | 4.207 | 4.219 | 4.231 | |
| 8. | 4.243 | 4.254 | 4.266 | 4.278 | 4.290 | 4.301 | 4.313 | 4.324 | 4.336 | 4.347 | |
| 9. | 4.359 | 4.370 | 4.382 | 4.393 | 4.405 | 4.416 | 4.427 | 4.438 | 4.450 | 4.461 | 11 |
| 20. | 4.472 | 4.483 | 4.494 | 4.506 | 4.517 | 4.528 | 4.539 | 4.550 | 4.561 | 4.572 | |
| 1. | 4.583 | 4.593 | 4.604 | 4.615 | 4.626 | 4.637 | 4.648 | 4.658 | 4.669 | 4.680 | |
| 2. | 4.690 | 4.701 | 4.712 | 4.722 | 4.733 | 4.743 | 4.754 | 4.764 | 4.775 | 4.785 | |
| 3. | 4.796 | 4.806 | 4.817 | 4.827 | 4.837 | 4.848 | 4.858 | 4.868 | 4.879 | 4.889 | 10 |
| 4. | 4.899 | 4.909 | 4.919 | 4.930 | 4.940 | 4.950 | 4.960 | 4.970 | 4.980 | 4.990 | |
| 25. | 5.000 | 5.010 | 5.020 | 5.030 | 5.040 | 5.050 | 5.060 | 5.070 | 5.079 | 5.089 | |
| 6 | 5.099 | 5.109 | 5.119 | 5.128 | 5.138 | 5.148 | 5.158 | 5.167 | 5.177 | 5.187 | |
| 7. | 5.196 | 5.206 | 5.215 | 5.225 | 5.235 | 5.244 | 5.254 | 5.263 | 5.273 | 5.282 | |
| 8. | 5.292 | 5.301 | 5.310 | 5.320 | 5.329 | 5.339 | 5.348 | 5.357 | 5.367 | 5.376 | 9 |
| 9. | 5.385 | 5.394 | 5.404 | 5.413 | 5.422 | 5.431 | 5.441 | 5.450 | 5.459 | 5.468 | |
| 30. | 5.477 | 5.486 | 5.495 | 5.505 | 5.514 | 5.523 | 5.532 | 5.541 | 5.550 | 5.559 | |
| 1. | 5.568 | 5.577 | 5.586 | 5.595 | 5.604 | 5.612 | 5.621 | 5.630 | 5.639 | 5.648 | |
| 2. | 5.657 | 5.666 | 5.675 | 5.683 | 5.692 | 5.701 | 5.710 | 5.718 | 5.727 | 5.736 | |
| 3. | 5.745 | 5.753 | 5.762 | 5.771 | 5.779 | 5.788 | 5.797 | 5.805 | 5.814 | 5.822 | |
| 4. | 5.831 | 5.840 | 5.848 | 5.857 | 5.865 | 5.874 | 5.882 | 5.891 | 5.899 | 5.908 | 8 |
| 35. | 5.916 | 5.925 | 5.933 | 5.941 | 5.950 | 5.958 | 5.967 | 5.975 | 5.983 | 5.992 | |
| 6. | 6.000 | 6.008 | 6.017 | 6.025 | 6.033 | 6.042 | 6.050 | 6.058 | 6.066 | 6.075 | |
| 7. | 6.083 | 6.091 | 6.099 | 6.107 | 6.116 | 6.124 | 6.132 | 6.140 | 6.148 | 6.156 | |
| 8. | 6.164 | 6.173 | 6.181 | 6.189 | 6.197 | 6.205 | 6.213 | 6.221 | 6.229 | 6.237 | |
| 9. | 6.245 | 6.253 | 6.261 | 6.269 | 6.277 | 6.285 | 6.293 | 6.301 | 6.309 | 6.317 | |
| 40. | 6.325 | 6.332 | 6.340 | 6.348 | 6.356 | 6.364 | 6.372 | 6.380 | 6.387 | 6.395 | |
| 1. | 6.403 | 6.411 | 6.419 | 6.427 | 6.434 | 6.442 | 6.450 | 6.458 | 6.465 | 6.473 | |
| 2. | 6.481 | 6.488 | 6.496 | 6.504 | 6.512 | 6.519 | 6.527 | 6.535 | 6.542 | 6.550 | |
| 3. | 6.557 | 6.565 | 6.573 | 6.580 | 6.588 | 6.595 | 6.603 | 6.611 | 6.618 | 6.626 | |
| 4. | 6.633 | 6.641 | 6.648 | 6.656 | 6.663 | 6.671 | 6.678 | 6.686 | 6.693 | 6.701 | |
| 45. | 6.708 | 6.716 | 6.723 | 6.731 | 6.738 | 6.745 | 6.753 | 6.760 | 6.768 | 6.775 | |
| 6. | 6.782 | 6.790 | 6.797 | 6.804 | 6.812 | 6.819 | 6.826 | 6.834 | 6.841 | 6.848 | |
| 7. | 6.856 | 6.863 | 6.870 | 6.877 | 6.885 | 6.892 | 6.899 | 6.907 | 6.914 | 6.921 | |
| 8. | 6.928 | 6.935 | 6.943 | 6.950 | 6.957 | 6.964 | 6.971 | 6.979 | 6.986 | 6.993 | |
| 9. | 7.000 | 7.007 | 7.014 | 7.021 | 7.029 | 7.036 | 7.043 | 7.050 | 7.057 | 7.064 | |

## Square Roots of Certain Fractions.

| N | $\sqrt{N}$ | N | $\sqrt{N}$ | N | $\sqrt{N}$ | N | $\sqrt{N}$ | N | $\sqrt{N}$ | N | $\sqrt{N}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1/2 | 0.7071 | 3/5 | 0.7746 | 4/7 | 0.7559 | 1/9 | 0.3333 | 5/12 | 0.6455 | 9/16 | 0.7500 |
| 1/3 | 0.5774 | 4/5 | 0.8944 | 5/7 | 0.8452 | 2/9 | 0.4714 | 7/12 | 0.7638 | 1 1/16 | 0.8292 |
| 2/3 | 0.8165 | 1/6 | 0.4082 | 6/7 | 0.9258 | 4/9 | 0.6667 | 1 1/12 | 0.9574 | 1 3/16 | 0.9014 |
| 1/4 | 0.5000 | 5/6 | 0.9129 | 1/8 | 0.3536 | 5/9 | 0.7454 | 1/16 | 0.2500 | 1 5/16 | 0.9682 |
| 3/4 | 0.8660 | 1/7 | 0.3780 | 3/8 | 0.6124 | 7/9 | 0.8819 | 3/16 | 0.4330 | 1/32 | 0.1768 |
| 1/5 | 0.4472 | 2/7 | 0.5345 | 5/8 | 0.7906 | 8/9 | 0.9428 | 5/16 | 0.5590 | 1/64 | 0.1250 |
| 2/5 | 0.6325 | 3/7 | 0.6547 | 7/8 | 0.9354 | 1/12 | 0.2887 | 7/16 | 0.6614 | 1/50 | 0.1414 |

## Square Roots.

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Avg. diff. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 50. | 7.071 | 7.078 | 7.085 | 7.092 | 7.099 | 7.106 | 7.113 | 7.120 | 7.127 | 7.134 | 7 |
| 1. | 7.141 | 7.148 | 7.155 | 7.162 | 7.169 | 7.176 | 7.183 | 7.190 | 7.197 | 7.204 | |
| 2. | 7.211 | 7.218 | 7.225 | 7.232 | 7.239 | 7.246 | 7.253 | 7.259 | 7.266 | 7.273 | |
| 3. | 7.280 | 7.287 | 7.294 | 7.301 | 7.308 | 7.314 | 7.321 | 7.328 | 7.335 | 7.342 | |
| 4. | 7.348 | 7.355 | 7.362 | 7.369 | 7.376 | 7.382 | 7.389 | 7.396 | 7.403 | 7.409 | |
| 55. | 7.416 | 7.423 | 7.430 | 7.436 | 7.443 | 7.450 | 7.457 | 7.463 | 7.470 | 7.477 | |
| 6. | 7.483 | 7.490 | 7.497 | 7.503 | 7.510 | 7.517 | 7.523 | 7.530 | 7.537 | 7.543 | |
| 7. | 7.550 | 7.556 | 7.563 | 7.570 | 7.576 | 7.583 | 7.589 | 7.596 | 7.603 | 7.609 | |
| 8. | 7.616 | 7.622 | 7.629 | 7.635 | 7.642 | 7.649 | 7.655 | 7.662 | 7.668 | 7.675 | |
| 9. | 7.681 | 7.688 | 7.694 | 7.701 | 7.707 | 7.714 | 7.720 | 7.727 | 7.733 | 7.740 | 6 |
| 60. | 7.746 | 7.752 | 7.759 | 7.765 | 7.772 | 7.778 | 7.785 | 7.791 | 7.797 | 7.804 | |
| 1. | 7.810 | 7.817 | 7.823 | 7.829 | 7.836 | 7.842 | 7.849 | 7.855 | 7.861 | 7.868 | |
| 2. | 7.874 | 7.880 | 7.887 | 7.893 | 7.899 | 7.906 | 7.912 | 7.918 | 7.925 | 7.931 | |
| 3. | 7.937 | 7.944 | 7.950 | 7.956 | 7.962 | 7.969 | 7.975 | 7.981 | 7.987 | 7.994 | |
| 4. | 8.000 | 8.006 | 8.012 | 8.019 | 8.025 | 8.031 | 8.037 | 8.044 | 8.050 | 8.056 | |
| 65. | 8.062 | 8.068 | 8.075 | 8.081 | 8.087 | 8.093 | 8.099 | 8.106 | 8.112 | 8.118 | |
| 6. | 8.124 | 8.130 | 8.136 | 8.142 | 8.149 | 8.155 | 8.161 | 8.167 | 8.173 | 8.179 | |
| 7. | 8.185 | 8.191 | 8.198 | 8.204 | 8.210 | 8.216 | 8.222 | 8.228 | 8.234 | 8.240 | |
| 8. | 8.246 | 8.252 | 8.258 | 8.264 | 8.270 | 8.276 | 8.283 | 8.289 | 8.295 | 8.301 | |
| 9. | 8.307 | 8.313 | 8.319 | 8.325 | 8.331 | 8.337 | 8.343 | 8.349 | 8.355 | 8.361 | |
| 70. | 8.367 | 8.373 | 8.379 | 8.385 | 8.390 | 8.396 | 8.402 | 8.408 | 8.414 | 8.420 | |
| 1. | 8.426 | 8.432 | 8.438 | 8.444 | 8.450 | 8.456 | 8.462 | 8.468 | 8.473 | 8.479 | |
| 2. | 8.485 | 8.491 | 8.497 | 8.503 | 8.509 | 8.515 | 8.521 | 8.526 | 8.532 | 8.538 | |
| 3. | 8.544 | 8.550 | 8.556 | 8.562 | 8.567 | 8.573 | 8.579 | 8.585 | 8.591 | 8.597 | |
| 4. | 8.602 | 8.608 | 8.614 | 8.620 | 8.626 | 8.631 | 8.637 | 8.643 | 8.649 | 8.654 | |
| 75. | 8.660 | 8.666 | 8.672 | 8.678 | 8.683 | 8.689 | 8.695 | 8.701 | 8.706 | 8.712 | |
| 6. | 8.718 | 8.724 | 8.729 | 8.735 | 8.741 | 8.746 | 8.752 | 8.758 | 8.764 | 8.769 | |
| 7. | 8.775 | 8.781 | 8.786 | 8.792 | 8.798 | 8.803 | 8.809 | 8.815 | 8.820 | 8.826 | |
| 8. | 8.832 | 8.837 | 8.843 | 8.849 | 8.854 | 8.860 | 8.866 | 8.871 | 8.877 | 8.883 | |
| 9. | 8.888 | 8.894 | 8.899 | 8.905 | 8.911 | 8.916 | 8.922 | 8.927 | 8.933 | 8.939 | |
| 80. | 8.944 | 8.950 | 8.955 | 8.961 | 8.967 | 8.972 | 8.978 | 8.983 | 8.989 | 8.994 | |
| 1. | 9.000 | 9.006 | 9.011 | 9.017 | 9.022 | 9.028 | 9.033 | 9.039 | 9.044 | 9.050 | |
| 2. | 9.055 | 9.061 | 9.066 | 9.072 | 9.077 | 9.083 | 9.088 | 9.094 | 9.099 | 9.105 | 5 |
| 3. | 9.110 | 9.116 | 9.121 | 9.127 | 9.132 | 9.138 | 9.143 | 9.149 | 9.154 | 9.160 | |
| 4. | 9.165 | 9.171 | 9.176 | 9.182 | 9.187 | 9.192 | 9.198 | 9.203 | 9.209 | 9.214 | |
| 85. | 9.220 | 9.225 | 9.230 | 9.236 | 9.241 | 9.247 | 9.252 | 9.257 | 9.263 | 9.268 | |
| 6. | 9.274 | 9.279 | 9.284 | 9.290 | 9.295 | 9.301 | 9.306 | 9.311 | 9.317 | 9.322 | |
| 7. | 9.327 | 9.333 | 9.338 | 9.343 | 9.349 | 9.354 | 9.359 | 9.365 | 9.370 | 9.375 | |
| 8. | 9.381 | 9.386 | 9.391 | 9.397 | 9.402 | 9.407 | 9.413 | 9.418 | 9.423 | 9.429 | |
| 9. | 9.434 | 9.439 | 9.445 | 9.450 | 9.455 | 9.460 | 9.466 | 9.471 | 9.476 | 9.482 | |
| 90. | 9.487 | 9.492 | 9.497 | 9.503 | 9.508 | 9.513 | 9.518 | 9.524 | 9.529 | 9.534 | |
| 1. | 9.539 | 9.545 | 9.550 | 9.555 | 9.560 | 9.566 | 9.571 | 9.576 | 9.581 | 9.586 | |
| 2. | 9.592 | 9.597 | 9.602 | 9.607 | 9.612 | 9.618 | 9.623 | 9.628 | 9.633 | 9.638 | |
| 3. | 9.644 | 9.649 | 9.654 | 9.659 | 9.664 | 9.670 | 9.675 | 9.680 | 9.685 | 9.690 | |
| 4. | 9.695 | 9.701 | 9.706 | 9.711 | 9.716 | 9.721 | 9.726 | 9.731 | 9.737 | 9.742 | |
| 95. | 9.747 | 9.752 | 9.757 | 9.762 | 9.767 | 9.772 | 9.778 | 9.783 | 9.788 | 9.793 | |
| 6. | 9.798 | 9.803 | 9.808 | 9.813 | 9.818 | 9.823 | 9.829 | 9.834 | 9.839 | 9.844 | |
| 7. | 9.849 | 9.854 | 9.859 | 9.864 | 9.869 | 9.874 | 9.879 | 9.884 | 9.889 | 9.894 | |
| 8. | 9.899 | 9.905 | 9.910 | 9.915 | 9.920 | 9.925 | 9.930 | 9.935 | 9.940 | 9.945 | |
| 9. | 9.950 | 9.955 | 9.960 | 9.965 | 9.970 | 9.975 | 9.980 | 9.985 | 9.990 | 9.995 | |

$\sqrt{\pi} = 1.77245+$ $\quad 1/\sqrt{\pi} = 0.56419$ $\quad \sqrt{\pi/2} = 1.25331$ $\quad \sqrt{e} = 1.64872$

NOTE. This page and the three that precede it are taken from Professor L. S. Marks's *Mechanical Engineers' Handbook*, published by McGraw-Hill Book Company, Inc.

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$\dot{e}_\theta = \dot{\theta}\hat{e}_\theta$

$\dot{e}_\theta = -\dot{\theta}\hat{e}_r$

$\bar{F} = -\frac{GMm}{r^2}\hat{r} = -\frac{GMm}{r^3}\bar{r}$

$\bar{F} = \bar{f}m$

$V = -\frac{GMm}{r} = \varphi m$

$\varphi = -\frac{GM}{r}$

$F = -\nabla V = -\nabla\varphi$

$V_A = \int_A^O \bar{F}\cdot dr$

$m_i \bar{v}_i = \bar{p}_i$

$\bar{r}_i \times \bar{p}_i = \bar{J}_i$

$J = \sum_{i=1}^{n} \bar{r}_i \times \bar{p}_i$

$\bar{L} = \sum_{i} \bar{r}_i \times \bar{F}_i$

AFQ
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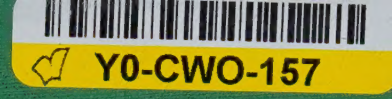
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